

Finsleroid Corrects Pressure and Energy of Universe. Respective Cosmological Equations

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Abstract

The Hubble constant proves to be the pseudo-Finsleroid–Landsberg factor. The covariantly conserved pseudo-Finsleroid–gravitational tensor is explicitly found after evaluating the respective Finsleroid–case curvature tensor and required contractions in attentive way. The equations arisen involve one parameter g of extension which measures the Finslerian deviation of the curvature of the indicatrix of unit vectors. The vector field $b^i(x)$ of the axes of the pseudo-Finsleroids is naturally identified to the field of average velocity vectors of matter of the universe. The consistent (and unique) continuation of the Robertson–Walker metric, and hence the Friedmann metrics, in the Finslerian domain with respect to the parameter g is arisen. The cosmological pressure and energy density prove to be linear functions of g^2 , so that the presence of the negative pressure seems to be not necessary to get the agreement with the observed negative nature of deceleration parameter. We clarify the explicit structure of all the involved tensorial objects.

Key words: Cosmological equations, energy density, pressure, Finsleroid geometry, relativity.

1. Introduction

To pattern the physics of the universe in terms of Finslerian as well as Riemannian equations, we are to comply with the observation evidence that to a great extent the universe is homogeneous and isotropic. It is well-known that the evidence is perfectly matched to the Friedmann-Robertson-Walker metrics formulated against the (pseudo-)Riemannian geometrical background (see [1,2]).

It is amazing, however, that the metrics can straightforwardly be extended to the (pseudo-)Finsleroid—geometry framework, strictly preserving both the homogeneity and isotropy. The Finsleroid charge g is the parameter of the extension.

Having had the far-developed Finsleroid geometry at hands, we can follow a new and lifted geometric route to search for due extensions of equations of the cosmological theory. What might happen if the cosmological value of the parameter g is essential, what if $g = 2$? The pending answer in general is: the cosmological “puzzles” may occur to be not manifestations of some new and exotic matter constituents, but merely implications of the Finsleroid—extensions of field equations, — at least contributions from such extensions are to be accounted for.

The nearest and simple consequence of the Finsleroid—case cosmological equations is that they may allow for the presence of the negative deceleration parameter without assuming the negative pressure. The systematic description of the observed phenomenon of the accelerated expansion of the universe can be found in the modern review [2].

In the Finsleroid—geometry context, the phenomenon may merely communicate us of the presence of the pseudo—Finsleroid charge g . Indeed, whenever the three—dimensional space is flat the conventional cosmology theory proposes the pressure

$$p_{\text{Friedmann}} = -(1 - 2q_{\text{cosm}}) H^2$$

(normalized by $8\pi G$), where H is the Hubble constant; by q_{cosm} we denote the deceleration parameter. The Finsleroid—case pressure

$$p = - \left(1 - 2q_{\text{cosm}} + \frac{g^2}{4} + g^2 q_{\text{cosm}} \right) H^2$$

(see (1.54)) provides us with the extension which involves the square g^2 of the pseudo—Finsleroid charge. One may conclude that

$$(2 - g^2)q_{\text{cosm}} = 1 + \frac{g^2}{4}, \quad \text{whenever } p = 0.$$

If we tentatively put here $q_{\text{cosm}} = -1$, we obtain $|g| = 2$. Thus, the negative value of the deceleration parameter q_{cosm} may well be compatible with the zero pressure if the Finsleroid charge g is not postulated to be zero.

The energy density of the universe also obtains corrections with respect to the square g^2 (see (1.56)).

The principal distinction of the Finsleroid geometry (generally, of the Finsler geometry) from the Riemannian geometry may be seen in the circumstance that the associated Finslerian metric tensor $g_{mn}(x, y)$ becomes being dependent on the pair (x, y) , — the so-called *line element*, — and hence on the tangent vectors y (which are supported by x). Such a dependence is succeeded to the curvature tensor $R_n^i{}_{mj}(x, y)$ and, after that, to the concomitant tensors and vectors

$$\rho_{ij} = \rho_{ij}(x, y), \quad \rho^i = \rho^i(x, y)$$

which arise upon extension of their relativistic gravitational precursors.

On the other hand, the deviation equation (1.30), as well as the geodesic equation (1.29), obviously belongs to the *Category of Observable Concepts*. In these equations, the vector y has the meaning of the velocity of a test particle (which does not contribute to the local gravitational field). Therefore, the variable y can be given the interpretation of the vector of a local observer located at the point x from which the vector y is issued. Following this motivation, we should interpret the dependence of $\{\rho_{ij}, \rho^i\}$ on y to be dependence on the velocity vector of the motion state of a local test-observer.

At the same time, in the cosmological pattern we have globally the *cosmological reference frame* produced by the field $b_{\text{cosm}}^i(x)$ of average velocity vectors of matter of the universe. Geometrically, the pseudo-Finsleroid is a rotund (axial-symmetric) body. Denote by $b_{\text{pseudo-Finsleroid}}^i(x)$ the vector field which is such that at any given point x the vector $b_{\text{pseudo-Finsleroid}}^i(x)$ assigns the axis of the pseudo-Finsleroid. It is quite natural to apply the identification

$$b_{\text{cosm}}^i(x) = b_{\text{pseudo-Finsleroid}}^i(x) \equiv b^i(x)$$

when starting to initiate the Finsleroid-extension of cosmological theory.

Accordingly, we shall comply with the following principle.

The Osculation–Correspondence Principle. The x -dependent extensions, which are functions of the points x of the underlying manifold, stem from their (x, y) -dependent counterparts through the osculation along the fundamental vector field $b^h(x)$:

$$y^h \rightarrow b^h(x).$$

The fields thus appeared are called *osculating* to their (x, y) -dependent counterparts along the vector field $b^h(x)$.

In particular, the field $\rho_{\{b\}}^i(x) := \rho_{|_{y^h=b^h}}^i$ (see (1.39)) is osculating to the field $\rho^i(x, y)$. In this equality, the left-hand part is osculating to the right-hand part along the vector field $b^h(x)$. The equality $g_{ij}(x, b^h(x)) = a_{ij}(x)$ (see (A.29) in Appendix A) means: the associated Riemannian metric tensor $a_{ij}(x)$ is osculating to the Finsleroid metric tensor $g_{ij}(x, y)$ along the vector field $b^h(x)$. The following important equality holds:

$$g_{ij}(x, y)|_{g=0} = g_{ij}(x, b^h(x)).$$

We can naturally interpret the osculating-cosmological fields as the *hydrodynamic* fields, that is, the fields which describe the universe at the hydrodynamic level of comprehension. In this vein, the associated Riemannian metric tensor $a_{ij}(x)$ is the hydrodynamic-level metric tensor of the universe. The pressure $p(x)$ used above is arisen upon the osculating-hydrodynamic method from the general (x, y) -dependent field (see (C.67) in Appendix C), — the same novelty should be said about the energy density $\rho(x)$ (see (C.66) in Appendix C).

It is a striking fact that the form (1.40) of the continuity equation is not affected by the Finsleroid extension (see also Proposition 1.4).

When pondering upon possibility to extend the cosmology theory in a Finsler way, the robust care should exercised that the general-status Finsler geometry (see [3-10]) does not provides us with a direct possibility to extend the gravitational field equations. This particular circumstance is dramatically at variance with what we have in the framework

of the Riemannian geometry. Indeed, in the latter geometry, the Riemannian curvature tensor, — to be denoted by $a_i^j{}_{mn}$, — produces uniquely the Riemannian Ricci tensor $a_i^j{}_{jn}$ on contraction. When one uses the last tensor to construct the gravitational field tensor $E^{\{\text{Riem}\}}_{in} = a_i^j{}_{jn} - \frac{1}{2}a_{in}a^{mk}{}_{mk}$, one may get the implication from appropriate identities obeyed by the curvature tensor $a_i^j{}_{mn}$ that the tensor $E^{\{\text{Riem}\}}_{in}$ is covariantly conserved: $\nabla_i E^{\{\text{Riem}\}}_n{}^i = 0$, where ∇_i stands for the Riemannian covariant derivative. The general-status Finsler geometry does not provide us with a similar lucky implication. In the Riemannian geometry, the equality $a_i^j{}_{jn} = a^j{}_{inj}$ holds, but in the Finsler geometry we are faced with $R_i^j{}_{jn} \neq R^j{}_{inj}$. Moreover, neither the contraction $R_i^j{}_{jn}$ nor the contraction $R^j{}_{inj}$ is symmetric with respect to the indices i, n , so that the Ricci tensor of the Riemannian geometry cannot be unambiguously extended to the domain of general Finsler geometry.

However, in the Finsleroid–Landsberg space applied in the present paper we have at our disposal the tensor ρ_{ij} (defined by (1.31)) which is covariantly-conserved (see (1.32)) and, therefore, is attractive to be used as a required extension of the gravitational tensor $E^{\{\text{Riem}\}}_{in}$. The entailed field $\rho^i = \rho^i{}_j y^j$ (see (1.33) and (1.34)) can naturally be treated as the current of energy.

The Finsleroid-based analysis involves a long chain of calculations, which are simple but sometimes tedious. The very method, however, can be explained in simple words.

Namely, let us start with a positive-definite Riemannian space R_N of the dimension N , referred to local coordinate set $\{x^i\}$. Denote by a_{mn} the Riemannian metric tensor of the space R_N and by ∇_n the associated Riemannian covariant derivative. Assume that a covariant vector field $b_i(x)$ be given on the space R_N . Set forth the unit length condition

$$a_{mn}b^mb^n = 1, \quad (1.1)$$

where $b^m = a^{mn}b_n$ is the associated contravariant vector field. We want to specify the covariant derivative $\nabla_n b_i$ in a manner attractive to apply in cosmological consideration. Let us choose the possibility

$$\nabla_n b_m = k(a_{mn} - b_m b_n), \quad (1.2)$$

where

$$k = k(x) \quad (1.3)$$

is a scalar. From (1.2) we conclude that the 1-form $b = b_i(x)dx^i$ must be closed:

$$\partial_i b_j - \partial_j b_i = 0, \quad (1.4)$$

where $\partial_i = \partial/\partial x^i$. From (1.1) and (1.2) it follows that the field $b^i(x)$ is geodesic:

$$b^n \nabla_n b_m = 0. \quad (1.5)$$

It is convenient to introduce the *projection tensor*

$$r_{mn} = a_{mn} - b_m b_n, \quad (1.6)$$

which fulfills the identities

$$b^m r_{mn} = b^n r_{mn} = 0. \quad (1.7)$$

In terms of the tensor (1.6) the condition (1.2) reads merely

$$\nabla_n b_m = k r_{mn}. \quad (1.8)$$

We shall also use the notation

$$k_n = \partial_n k \quad (1.9)$$

and

$$(bk) = b^n k_n, \quad (1.10)$$

together with

$$\tilde{k}_n = \partial_n k + k^2 b_n \quad (1.11)$$

and

$$(b\tilde{k}) = b^n \tilde{k}_n. \quad (1.12)$$

If we introduce in the space \mathcal{R}_N a *b-geodesic coordinate set* $\{z^A\}$, $A = \{a, N\}$, $a, b, c, d = 1, \dots, N-1$, arisen upon an admissible coordinate transformation $x^i = x^i(z^A)$ such that the vector field $b^i(x)$ occurs featuring tangent to the z^N -coordinate line, then it is easy to see that with respect to the coordinates $\{z^A\}$

$$b^N = 1, \quad b_a = 0, \quad a_{NN} = 1, \quad a_{Na} = 0, \quad r_{NN} = 0, \quad r_{Na} = 0, \quad (1.13)$$

and the squared Riemannian metric $(ds)^2 = a_{mn} dx^m dx^n$ is transformed to the sum

$$(ds)^2 = (dz^N)^2 + r_{ab}(z) dz^a dz^b. \quad (1.14)$$

Invoke the *factorization condition*

$$r_{ab}(z^A) = (\phi(z^A))^2 p_{ab}(z^c), \quad (1.15)$$

so that the tensor $p_{ab}(z^c)$ is assumed to be independent of the coordinate z^N . By rewriting the condition (1.8) in terms of the coordinate $\{z^A\}$ it is easy to see that the factor k entering (1.8) is obtainable through

$$k = \frac{1}{\phi} \frac{\partial \phi}{\partial z^N}. \quad (1.16)$$

Let us set forth also the condition

$$\frac{\partial \phi(z^A)}{\partial z^a} = 0,$$

that is,

$$\phi = \phi(z^N). \quad (1.17)$$

Converting this condition to the initial coordinates x^i yields

$$k_n = (bk)b_n, \quad (1.18)$$

which entails the representation

$$\tilde{k}_n = (b\tilde{k})b_n \quad (1.19)$$

for the vector (1.11). With postulating (1.15), it is convenient to use the notation

$$\dot{k} = \frac{dk}{dz^N}, \quad (1.20)$$

getting

$$\frac{\dot{\phi}}{\phi} = k, \quad \dot{k} + k^2 = \frac{1}{\phi} \ddot{\phi}, \quad (1.21)$$

together with

$$(bk) = \dot{k} \quad (1.22)$$

and

$$(b\tilde{k}) = \dot{k} + k^2. \quad (1.23)$$

Thus the following proposition is a truth.

Proposition 1.1. *When in addition to*

$$a_{mn}b^mb^n = 1, \quad \nabla_n b_m = kr_{mn}, \quad k_n = (bk)b_n,$$

the factorization condition (1.15) is applied, the Riemannian space R_N is of the warped type

$$(ds)^2 = (dz^N)^2 + (\phi(z^N))^2 p_{ab}(z^c) dz^a dz^b. \quad (1.24)$$

Definition. Under the conditions of Proposition 1.1 we say that we deal with the *special case* of the Riemannian space \mathcal{R}_N .

Note. Obviously, the conditions (1.17) and (1.18) are equivalent, so that the special case can equivalently be characterized by the condition (1.18) instead of the condition (1.17).

To lift the consideration to the Finsler-geometry level with the aim of specifying an appropriate Finslerian metric function $K(x, y)$, where y are tangent vectors, we assume that the sought function $K(x, y)$ be of the functional dependence

$$K(x, y) = \Phi^{PD} \left(g(x), b_i(x), a_{ij}(x), y \right), \quad (1.25)$$

where $g(x)$ is a scalar playing the role of the *Finslerian parameter* of extension and Φ is a scalar which smoothness is to be of at least class C^4 (if the class C^∞ is not realized).

FINSLEROID PRINCIPLE OF EXTENSION. In each tangent space, the indicatrix (=surface of unit-length vectors defined by the function K) is to be of constant curvature. The indicatrix is rotund with the vector $b^i = a^{ij}b_j$ being the axial symmetry. The indicatrix is closed and everywhere regular and the resultant Finsler space is positive-definite.

Under these conditions, we call the unit ball bounded by the indicatrix the *Finsleroid*. For the curvature \mathcal{R} of the indicatrix, the value

$$\mathcal{R}_{\text{Finsleroid Indicatrix}} = 1 - \frac{g^2}{4} \quad (1.26)$$

is obtained. Thereby the metric function K together with the entailed \mathcal{FF}_g^{PD} -space (the meaning of the upperscripts PD reads “positive-definite”) is completely and uniquely determined (see the formulas (A.1)–(A.16) in Appendix A). Notice that the curvature value (1.26) is independent of neither the tensor $a_{ij}(x)$ nor the field $b_i(x)$.

The continuation parameter g is dimensionless, being of *pure-geometrical meaning*.

CRUCIAL DEFINITION. The \mathcal{FF}_g^{PD} -space subjected to the conditions $\nabla_n b_m = kr_{mn}$ and

$$g = \text{const} \quad (1.27)$$

is called the *Finsleroid–Landsberg space*. The scalar k is called the *Finsleroid–Landsberg factor*.

In any Finsler space (see the books [3-5]), the geodesic equations are defined in the unique way by the help of the so-called spray coefficients

$$G^i := \gamma^i_{mn}(x, y)y^m y^n \quad (1.28)$$

with γ^i_{mn} standing for the Finslerian Christoffel symbols constructed from the Finsleroid–Finsler metric function K . The geodesic equations read

$$\frac{d^2 x^i}{d\sigma^2} + G^i(x, y) = 0, \quad y^h = \frac{dx^h}{d\sigma}, \quad (1.29)$$

where $d\sigma = \sqrt{|g_{mn}(x, dx)dx^m dx^n|}$ is the Finslerian arc-length.

The geodesic deviation equation reads

$$\frac{\delta^2 \eta^i}{d\sigma^2} + R_n{}^i(x, y)\eta^n = 0, \quad y^h = \frac{dx^h}{d\sigma}. \quad (1.30)$$

The deviation tensor $R_n{}^i(x, y)$ thus appeared entails the full curvature tensor $R_n{}^i{}_{mj}(x, y)$ upon applying appropriate differentiation with respect to the variable y (see (A.59)–(A.61) in Appendix A).

The known fact (indicated in [11-14]) is that *in the Finsleroid–Landsberg space the tensor*

$$\rho_{ij} := \frac{1}{2}(R_i{}^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm} \quad (1.31)$$

is covariantly conserved:

$$\rho^i{}_{j|i} \equiv 0. \quad (1.32)$$

The covariant derivative $|i$ is constructed according to the known Finslerian rule (see (A.68)–(A.70) in Appendix A).

Let us construct by means of contraction the *produced current*

$$\rho^i := \rho^i{}_j y^j. \quad (1.33)$$

Since $y^j|_n = 0$ in any Finsler space, from (1.32) it issues that the following proposition is valid.

Proposition 1.2. *In the Finsleroid–Landsberg space the current ρ^i is covariantly conserved:*

$$\rho^i{}_{|i} \equiv 0. \quad (1.34)$$

Definition. The \mathcal{FF}_g^{PD} -space is said to be of the *isotropic case* if the conditions claimed in Proposition 1.1 are fulfilled and the tensor p_{ab} entered the metric (1.24) is such that the Riemannian curvature tensor evaluated from the tensor corresponds to the constant-curvature case (that is, the representations (C.32)–(C.34) of Appendix C hold).

The current can be expanded with respect to the basis $\{b^i, y^i\}$

$$\rho^i = c_1(x, y)b^i + c_2(x, y)y^i \quad (1.35)$$

(the coefficients c_1, c_2 can be found from the formula (C.61) of Appendix C) or, alternatively, with respect to the basis $\{y^i, A^i\}$

$$\rho^i = n_1(x, y)y^i + n_2(x, y)A^i \quad (1.36)$$

(the coefficients n_1, n_2 can be found from the formula (C.58) of Appendix C), where A^i is the contraction of the Cartan tensor of the Finsler geometry (see (A.34)–(A.38) in Appendix A).

In the dimension

$$N = 4$$

from (C.68) we may conclude that the quantity

$$p := -\mathcal{P} \quad (1.37)$$

equals

$$p = -\left[\varkappa + \left(1 - \frac{g^2}{4}\right) k^2 + (2 + g^2)(b\tilde{k}) \right] \quad (1.38)$$

(the equality (C.44) has been used), and with the definition

$$\rho_{\{b\}}^i(x) := \rho_{|_{y^h=b^h}}^i \quad (1.39)$$

of the Finsleroid–extension of the current from (C.70) and (C.71) we obtain

$$\nabla_i \rho_{\{b\}}^i = -3kp. \quad (1.40)$$

In the space–time context proper, we take the dimension $N = 4$ and apply the pseudo-Finsleroid space. A close inspection shows that we may straightforwardly perform the convertation

$$K \rightarrow F \quad (1.41)$$

of the positive–definite Finsleroid metric function K into the relativistic–case pseudo-Finsleroid metric function, to be denoted by F , by applying the formal change

$$g \rightarrow ig \quad (1.42)$$

(where i is the imaginary unity), taking the associated metric tensor $a_{mn}(x)$ to be of the pseudo–Riemannian type

$$a_{mn}(x) = b_m(x)b_n(x) - r_{mn}(x) \quad (1.43)$$

with the condition of positive–definiteness

$$r_{mn}y^m y^n > 0 \quad \text{and} \quad \text{rank}(r_{mn}) = N - 1 \quad (1.44)$$

(we always assume $y \neq 0$). The covariant vector field $b_i(x)$ is to be taken time–like and the unit condition (1.1) persists. The indicatrix curvature value (1.26) is replaced by

$$\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} = -\left(1 + \frac{g^2}{4}\right). \quad (1.45)$$

The respective scalar $g(x)$ will be called the *pseudo-Finsleroid charge*. Similarly to (1.25), the pseudo-Finsleroid metric function F is of the functional structure

$$F(x, y) = \Phi^{SR} \left(g(x), b_i(x), a_{ij}(x), y \right). \quad (1.46)$$

We denote the resultant space by the \mathcal{FF}_g^{SR} -space (with the upperscripts SR meaning “special-relativistic”). The Landsberg condition preserves its form (1.8). With the Landsberg condition assumed to hold, we call the \mathcal{FF}_g^{SR} -space the *pseudo-Finsleroid-Landsberg space* and call the scalar k the *pseudo-Finsleroid-Landsberg factor*.

In the cosmological application, the vector field $b^i(x) = a^{ij}(x)b_j(x)$ of the pseudo-Finsleroid axes is naturally identified with the field of average velocity vectors of matter of the universe, thereby *the field $b^i(x)$ represents the co-moving reference frame of the universe*. Under these conditions, we call the covariantly conserved tensor ρ_{ij} given by (1.32) the *Finsleroid-gravitational tensor* and the vector field ρ^i obtained in the way (1.33) the *current of energy of the universe*.

In such a relativistic-cosmological physical case, we are to make in the warped-metric representation (1.24) the change

$$z^N \rightarrow z^0 = t, \quad \phi \rightarrow L, \quad (1.47)$$

with t meaning the *cosmological time*, and with

$$L = L(t) \quad (1.48)$$

getting the sense of the *cosmological scale factor*. Accordingly, the equalities (1.21) change to

$$\frac{\dot{L}}{L} = k, \quad \dot{k} + k^2 = \frac{1}{L} \ddot{L}, \quad (1.49)$$

so that the k acquires the meaning of the *Hubble constant*. Also,

$$(b\tilde{k}) = \dot{k} + k^2 = -\frac{q_{\text{cosm}}}{k^2}, \quad (1.50)$$

where q_{cosm} makes sense of the cosmological *deceleration parameter*. In the isotropic case we have

$$\varkappa = \frac{1}{L^2} \varkappa_1, \quad \varkappa_1 = -1, 0, 1 \quad (1.51)$$

(see (C.32)–(C.34) in Appendix C).

Under these circumstances, the metric (1.24) changes to read

$$(ds)^2 = (dt)^2 - (L(t))^2 p_{ab}(z^c) dz^a dz^b. \quad (1.52)$$

Thus we are arriving at

Proposition 1.3. *The Finsleroid-Landsberg space thus arisen is the (unique) continuation of the Robertson-Walker metric in the Finslerian domain. It is the Hubble constant H that plays the role of the pseudo-Finsleroid-Landsberg factor k :*

$$H = k. \quad (1.53)$$

Since the cosmological Friedmann metrics are particular cases of the Robertson-Walker metric, we also obtain due pseudo-Finsleroid-Landsberg continuations of the metrics.

In this way we arrive at the *Finsleroid hydrodynamic pressure*

$$p = - \left(1 - 2q_{\text{cosm}} + \frac{g^2}{4} + g^2 q_{\text{cosm}} \right) H^2 - \varkappa \quad (1.54)$$

(from (1.38)), and the *Finsleroid energy current*

$$\rho^i = \rho b^i \quad (1.55)$$

together with the *Finsleroid energy density*

$$\rho = 3 \left(1 + \frac{g^2}{4} \right) H^2 + 3\varkappa \quad (1.56)$$

(the formulas (C.64) and (C.66) of Appendix C have been used). From (1.40) we get the continuity equation

$$\nabla_i \rho^i = -3Hp. \quad (1.57)$$

Applying (A.109) and noting

$$b^i \partial_i \rho = \frac{d\rho}{dt}$$

together with

$$\nabla_i b^i = 3k \quad (1.58)$$

(consider the Landsberg-case condition (1.2) at $N = 4$), we may write the equation (1.57) in the form

$$\frac{d\rho}{dt} + 3H\rho = -3Hp. \quad (1.59)$$

We can also write this as

$$\frac{d(\rho L^3)}{dL} = -3pL^2, \quad (1.60)$$

which is precisely of the form characteristic of the respective Friedmann-case equation. The Finsleroid charge g does not enter the last equation.

Thus we are entitled to formulate

Proposition 1.4. *The Finsleroid-Landsberg extension of the Friedmann metrics does not change the conventional equation which describes the energy density of the universe. When the pressure vanishes, that is $p = 0$, we obtain from (1.60) the well-known Friedmann-case law*

$$\rho \sim \frac{1}{L^3}. \quad (1.61)$$

NOTE. In the conventional cosmology theory, the pressure is

$$p_{\text{Friedmann}} = - (1 - 2q_{\text{cosm}}) H^2 - \varkappa. \quad (1.62)$$

The Finsleroid-case pressure p given by (1.54) extends this version (1.62) by the terms which involve the square of the pseudo-Finsleroid charge, namely g^2 .

Whenever the three-dimensional space is flat:

$$\varkappa = 0, \quad (1.63)$$

from (1.56) we may conclude the validity of the law

$$\frac{\text{Finsleroid-density of energy}}{\text{Friedmann-density of energy}} = 1 + \frac{g^2}{4} \equiv -\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} \quad (1.64)$$

and the pressure (1.54) becomes such that

$$\frac{\text{Finsleroid pressure}}{\text{Friedmann pressure}} = \frac{1 - 2q_{\text{cosm}} + \frac{g^2}{4} + g^2 q_{\text{cosm}}}{1 - 2q_{\text{cosm}}} \quad (1.65)$$

When the special-case conditions (C.1)–(C.12) and the isotropy conditions (C.32)–(C.34), – set forth in Appendix C, – are fulfilled, the tensor ρ^{ik} can be evaluated in the form of the expansion with respect to the basis $\{g^{ik}, A^i, y^i\}$

$$\rho^{ik} = \frac{M_7}{2K^2} g^{ik} + \frac{M_8}{2K^2} \frac{1}{A_l A^l} A^i A^k - \frac{2q}{NKg} \frac{1}{K^2} \left(Y_{\{y\}} y^k A^i + Z_{\{e\}} A^k y^i \right) + \frac{M_{10}}{2K^4} y^i y^k \quad (1.66)$$

or with respect to the basis $\{a^{ik}, b^i, y^i\}$

$$\rho^{ik} = E_1 a^{ik} + E_2 b^i b^k + E_{3Y} b^i y^k + E_{3Z} b^k y^i + E_4 y^i y^k. \quad (1.67)$$

The involved coefficients $E_1, E_2, E_{3Y}, E_{3Z}, E_4$ depend not only on point x but also on tangent vectors y supported by the x . The explicit form of the coefficients can be found in Appendix D. Unless $g = 0$, the non-symmetry

$$E_{3Y} \neq E_{3Z}, \quad Y_{\{y\}} \neq Z_{\{e\}} \quad (1.68)$$

entailing

$$\rho_{ij} \neq \rho_{ji} \quad (1.69)$$

takes place. The skew part

$$\rho^{ik} - \rho^{ki}$$

is explicitly evaluated in Appendix E.

The coefficients E_{3Y} , E_{3Z} , and E_4 are proportional to the Finsleroid parameter g :

$$E_{3Y} = gT_{3Y}, \quad E_{3Z} = gT_{3Z}, \quad E_4 = gT_4 \quad (1.70)$$

(see (D.40), (D.42), and (D.45)). They disappear, therefore, in the Riemannian limit:

$$(E_{3Y})_{|g=0} = (E_{3Z})_{|g=0} = (E_4)_{|g=0} = 0. \quad (1.71)$$

We also have

$$(E_1)_{|g=0} = \mathcal{P} \quad (1.72)$$

(from (D.35) and (D.23)) and

$$(E_2)_{|g=0} = -(N-2)(\xi + (b\tilde{k})), \quad \xi = -k^2 - \varkappa, \quad (b\tilde{k}) = \dot{k} + k^2 \quad (1.73)$$

(from (D.36) and (D.27)).

The term Λg_{ij} with the cosmological constant Λ can also be added to the tensor ρ_{ij} not destroying the validity of the law of the covariant-divergence conservation.

Appendix A: Initial knowledge of the \mathcal{FF}_g^{PD} -objects

Let $\mathcal{R}_N = \{M, a_{mn}\}$ be a Riemannian space, where M is an N -dimensional manifold and a_{mn} is a positive-definite Riemannian metric tensor. We introduce on M a scalar field $g = g(x)$ subject to ranging

$$-2 < g(x) < 2, \quad (\text{A.1})$$

and apply the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \quad G = \frac{g}{h}. \quad (\text{A.2})$$

The *characteristic quadratic form*

$$B(x, y) := b^2 + gqb + q^2 \equiv \frac{1}{2} \left[(b + g_+q)^2 + (b + g_-q)^2 \right] > 0, \quad (\text{A.3})$$

where $g_+ = \frac{1}{2}g + h$ and $g_- = \frac{1}{2}g - h$, is of the negative discriminant

$$D_{\{B\}} = -4h^2 < 0 \quad (\text{A.4})$$

and, therefore, is positively definite. In the limit $g \rightarrow 0$, the definition (A.3) degenerates to the quadratic form of the input Riemannian metric tensor:

$$B|_{g=0} = a_{mn}y^my^n. \quad (\text{A.5})$$

Definition. The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)} J(x, y) \quad (\text{A.6})$$

and

$$J(x, y) = e^{-\frac{1}{2}G(x)f(x,y)}, \quad (\text{A.7})$$

where

$$f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if } b \geq 0, \quad (\text{A.8})$$

and

$$f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if } b \leq 0, \quad (\text{A.9})$$

with

$$L = q + \frac{g}{2}b, \quad (\text{A.10})$$

is called the *Finsleroid-Finsler metric function*.

The function K has been normalized such that

$$0 \leq f \leq \pi,$$

$$f = 0, \quad \text{if } q = 0 \quad \text{and } b > 0; \quad f = \pi, \quad \text{if } q = 0 \quad \text{and } b < 0,$$

and the Finsleroid length of the vector b^i is equal to 1:

$$K(x, b^i(x)) = 1, \quad (\text{A.11})$$

or

$$||b||_{\text{Finsleroid}} = 1.$$

Sometimes it is convenient to use also the function

$$A = b + \frac{g}{2}q. \quad (\text{A.12})$$

The identities

$$L^2 + h^2 b^2 = B, \quad A^2 + h^2 q^2 = B \quad (\text{A.13})$$

are valid.

The zero-vector $y = 0$ is excluded from consideration. The positive (not absolute) homogeneity holds:

$$K(x, \lambda y) = \lambda K(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y.$$

Definition. The arisen space

$$\mathcal{FF}_g^{PD} := \{\mathcal{R}_N; b_i(x); g(x); K(x, y)\} \quad (\text{A.14})$$

is called the *Finsleroid-Finsler space*.

Definition. The space \mathcal{R}_N entering the above definition is called the *associated Riemannian space*.

Definition. Within any tangent space $T_x M$, the Finsleroid-metric function $K(x, y)$ produces the *Finsleroid*

$$\mathcal{F}_{g\{x\}}^{PD} := \{y \in \mathcal{F}_g^{PD} : y \in T_x M, K(x, y) \leq 1\}. \quad (\text{A.15})$$

Definition. The *Finsleroid Indicatrix* $I_{g\{x\}}^{PD} \in T_x M$ is the boundary of the Finsleroid:

$$I_{g\{x\}}^{PD} := \{y \in \mathcal{F}_g^{PD} : y \in T_x M, K(x, y) = 1\}. \quad (\text{A.16})$$

Since at $g = 0$ the \mathcal{FF}_g^{PD} -space is Riemannian, then the body $\mathcal{F}_{g=0\{x\}}^{PD}$ is a unit ball and $I_{g=0\{x\}}^{PD}$ is a unit sphere.

Definition. The scalar $g(x)$ is called the *Finsleroid charge*. The 1-form $b = b_i(x)y^i$ is called the *Finsleroid-axis 1-form*.

Under these conditions, we can explicitly calculate from the function K the distinguished Finslerian tensors, and first of all the covariant tangent vector $\hat{y} = \{y_i\}$, the Finslerian metric tensor $\{g_{ij}\}$ together with the contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, and the angular metric tensor $\{h_{ij}\}$, by making use of the following conventional Finslerian rules in succession:

$$y_i := \frac{1}{2} \frac{\partial K^2}{\partial y^i}, \quad g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad h_{ij} := g_{ij} - y_i y_j \frac{1}{K^2}. \quad (\text{A.17})$$

It is convenient to use the variables

$$v^i := y^i - b b^i, \quad v_m := u_m - b b_m = r_{mn} y^n \equiv r_{mn} v^n \equiv a_{mn} v^n, \quad (\text{A.18})$$

where $r_{mn} = a_{mn} - b_m b_n$ is the projection tensor (1.6). Notice that

$$r^i_n := a^{im} r_{mn} = \delta^i_n - b^i b_n = \frac{\partial v^i}{\partial y^n}, \quad (\text{A.19})$$

$$v_i b^i = v^i b_i = 0, \quad r_{ij} b^j = r^i_j b^j = b_i r^i_j = 0 \quad (\text{A.20})$$

(cf. (1.7)), and

$$q = \sqrt{r_{ij} v^i v^j}, \quad (\text{A.21})$$

together with

$$\frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}, \quad \frac{\partial(b/q)}{\partial y^i} = \frac{2B}{NKgq^2} A_i, \quad (\text{A.22})$$

where A_i is the vector defined below in (A.34).

In terms of the variables (A.18) we obtain the representations

$$y_i = \left(v_i + (b + gq) b_i \right) \frac{K^2}{B}, \quad (\text{A.23})$$

$$g_{ij} = \left[a_{ij} + \frac{g}{B} \left(q(b + gq) b_i b_j + q(b_i v_j + b_j v_i) - b \frac{v_i v_j}{q} \right) \right] \frac{K^2}{B}, \quad (\text{A.24})$$

and the reciprocal components $(g^{ij}) = (g_{ij})^{-1}$ read

$$g^{ij} = \left[a^{ij} + \frac{g}{q} (b b^i b^j - b^i y^j - b^j y^i) + \frac{g}{Bq} (b + gq) y^i y^j \right] \frac{B}{K^2}, \quad (\text{A.25})$$

or

$$g^{ij} = \left[a^{ij} + \frac{g}{B} \left(-bq b^i b^j - q(b^i v^j + b^j v^i) + (b + gq) \frac{v^i v^j}{q} \right) \right] \frac{B}{K^2}. \quad (\text{A.26})$$

It is a useful exercise to verify that

$$g_{ij} g^{jn} = \delta_i^n. \quad (\text{A.27})$$

We have also

$$g_{ij}(x, y) \big|_{y=0} = a_{ij}(x) \quad (\text{A.28})$$

and

$$g_{ij}(x, b^n(x)) = a_{ij}(x), \quad (\text{A.29})$$

together with

$$y_i b^i = (b + gq) \frac{K^2}{B}, \quad g_{ij} b^j = \left(b_i + gq \frac{y_i}{K^2} \right) \frac{K^2}{B}, \quad (\text{A.30})$$

$$g_{ij} v^j = (S^2 v_i + gq^3 b_i) \frac{K^2}{B^2}, \quad (\text{A.31})$$

$$g^{ij} a_{ij} = \frac{NB + gq^2}{K^2}, \quad h_{ij} b^j = \left(b_i - b \frac{y_i}{K^2} \right) \frac{K^2}{B}. \quad (\text{A.32})$$

The determinant of the metric tensor is everywhere positive:

$$\det(g_{ij}) = \left(\frac{K^2}{B}\right)^N \det(a_{ij}) > 0. \quad (\text{A.33})$$

From the last expression we can explicate the vector

$$A_i := K \frac{\partial \ln \left(\sqrt{\det(g_{mn})} \right)}{\partial y^i}, \quad (\text{A.34})$$

obtaining

$$A_i = \frac{NK}{2} g \frac{1}{q} (b_i - \frac{b}{K^2} y_i), \quad (\text{A.35})$$

or

$$A_i = \frac{NK}{2} g \frac{1}{qB} (q^2 b_i - b v_i). \quad (\text{A.36})$$

Raising the index according to the general rule (namely $A^i = g^{ij} A_j$) yields

$$A^i = \frac{N}{2} g \frac{1}{qK} [B b^i - (b + gq) y^i], \quad (\text{A.37})$$

or

$$A^i = \frac{N}{2} g \frac{1}{qK} [q^2 b^i - (b + gq) v^i]. \quad (\text{A.38})$$

We have

$$A_i b^i = \frac{N}{2} g q \frac{K}{B}, \quad A^i b_i = \frac{N}{2} g q \frac{1}{K}, \quad (\text{A.39})$$

so that

$$A_h A^h = \frac{N^2}{4} g^2, \quad (\text{A.40})$$

After that, we can elucidate the algebraic structure of the associated Cartan tensor

$$A_{ijk} := \frac{K}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad (\text{A.41})$$

which leads to the following simple and remarkable result: the Cartan tensor associated with the Finsleroid–Finsler metric function K is of the following special algebraic form:

$$A_{ijk} = \frac{1}{N} \left(h_{ij} A_k + h_{ik} A_j + h_{jk} A_i - \frac{1}{A_h A^h} A_i A_j A_k \right). \quad (\text{A.42})$$

Since

$$\frac{v^i v^j}{q} \rightarrow 0 \quad \text{when} \quad v^i \rightarrow 0$$

(notice (A.18)) the components g_{ij} and g^{ij} given by (A.24) and (A.25) are smooth on all the slit tangent bundle $TM \setminus 0$. However, the components of the Cartan tensor are singular at $v^i = 0$, as this is apparent from the above formulas (A.35)–(A.36) in which the *pole singularity* takes place at $q = 0$. Therefore, *on the slit tangent bundle $TM \setminus 0$ the \mathcal{FF}_g^{PD} -space is smooth of the class C^2 and not of the class C^3 .*

We use the Riemannian metric tensor a_{ij} of the associated Riemannian space to construct the Riemannian Christoffel symbols

$$a^k_{ij} := \frac{1}{2} a^{kn} (\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \quad (\text{A.43})$$

$(\partial_j = \partial/\partial x^j)$ which give rise to the Riemannian curvature tensor

$$a_n^i{}_{km} = \frac{\partial a^i{}_{nm}}{\partial x^k} - \frac{\partial a^i{}_{nk}}{\partial x^m} + a^u{}_{nm}a^i{}_{uk} - a^u{}_{nk}a^i{}_{um} \quad (\text{A.44})$$

and the Riemannian covariant derivative

$$\nabla_i b_j := \partial_i b_j - b_k a^k{}_{ij} \quad (\text{A.45})$$

of the involved vector field $b_i(x)$. The vanishing

$$b^j \nabla_i b_j = 0 \quad (\text{A.46})$$

(the vector b_j is of the unit Riemannian length according to (1.1)) nullifies many terms which appear while performing various calculations.

According to the conclusions drawn in the previous work [11-12], the equality

$$\nabla_i b_j = k r_{ij} \quad (\text{A.47})$$

is characteristic of the property that the Finsleroid space be of the Landsberg type. We call the k thus appeared the *Finsleroid-Landsberg factor*.

From (A.47) we get the commutator

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) b_k = \tilde{k}_m (a_{nk} - b_n b_k) - \tilde{k}_n (a_{mk} - b_m b_k) = \tilde{k}_m r_{nk} - \tilde{k}_n r_{mk},$$

so that

$$b_j a_n^j{}_{km} = -\tilde{k}_k r_{nm} + \tilde{k}_m r_{nk} \quad (\text{A.48})$$

is valid with the vector

$$\tilde{k}_n = \frac{\partial k}{\partial x^n} + k^2 b_n. \quad (\text{A.49})$$

In turn, from (A.48) it follows that

$$(\nabla_i b_j) a_n^j{}_{km} + b_j \nabla_i a_n^j{}_{km} = -(\nabla_i \tilde{k}_k) r_{nm} + \tilde{k}_k \nabla_i (b_n b_m) + (\nabla_i \tilde{k}_m) r_{nk} - \tilde{k}_m \nabla_i (b_n b_k).$$

We get

$$b_j \nabla_i a_n^j{}_{km} = -k r_{ij} a_n^j{}_{km}$$

$$-(\nabla_i \tilde{k}_k) r_{nm} + k \tilde{k}_k (b_n r_{im} + b_m r_{in}) + (\nabla_i \tilde{k}_m) r_{nk} - k \tilde{k}_m (b_n r_{ik} + b_k r_{in}).$$

Applying here the cyclic identity yields

$$-k r_{ij} a_n^j{}_{km} - (\nabla_i \tilde{k}_k) r_{nm} + k \tilde{k}_k (b_n r_{im} + b_m r_{in}) + (\nabla_i \tilde{k}_m) r_{nk} - k \tilde{k}_m (b_n r_{ik} + b_k r_{in})$$

$$-k r_{kj} a_n^j{}_{mi} - (\nabla_k \tilde{k}_m) r_{ni} + k \tilde{k}_m (b_n r_{ki} + b_i r_{kn}) + (\nabla_k \tilde{k}_i) r_{nm} - k \tilde{k}_i (b_n r_{km} + b_m r_{kn})$$

$$-k r_{mj} a_n^j{}_{ik} - (\nabla_m \tilde{k}_i) r_{nk} + k \tilde{k}_i (b_n r_{mk} + b_k r_{mn}) + (\nabla_m \tilde{k}_k) r_{ni} - k \tilde{k}_k (b_n r_{mi} + b_i r_{mn}) = 0, \quad (\text{A.50})$$

which makes us conclude that the framework developed must be compatible with the identity

$$\nabla_m \tilde{k}_k - \nabla_k \tilde{k}_m + k(\tilde{k}_k b_m - \tilde{k}_m b_k) = 0. \quad (\text{A.51})$$

It is immediately seen that special case, as given by the warped representation (1.24) of the square $(ds)^2$, does obey the identity (A.51).

One considers the *Finslerian Christoffel symbols*

$$\gamma^k_{ij} := g^{kn} \gamma_{inj} \quad (\text{A.52})$$

with

$$\gamma_{inj} := \frac{1}{2}(\partial_j g_{ni} + \partial_i g_{nj} - \partial_n g_{ji}) \quad (\text{A.53})$$

and construct the induced *spray coefficients*

$$G^k = \gamma^k_{ij} y^i y^j. \quad (\text{A.54})$$

The second-degree positive homogeneity

$$G^k(x, \lambda y) = \lambda^2 G^k(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y, \quad (\text{A.55})$$

is valid. With these coefficients, we obtain the coefficients

$$G^i_k := \frac{\partial G^i}{\partial y^k}, \quad G^i_{km} := \frac{\partial G^i_k}{\partial y^m}, \quad G^i_{kmn} := \frac{\partial G^i_{km}}{\partial y^n}, \quad (\text{A.56})$$

and

$$\bar{G}^i = \frac{1}{2} G^i, \quad \bar{G}^i_k = \frac{1}{2} G^i_k, \quad \bar{G}^i_{km} = \frac{1}{2} G^i_{km}, \quad \bar{G}^i_{kmn} = \frac{1}{2} G^i_{kmn}. \quad (\text{A.57})$$

The homogeneity (A.55) entails the identities

$$2G^i = G^i_k y^k, \quad G^i_k = G^i_{km} y^m, \quad G^i_{kmn} y^n = 0. \quad (\text{A.58})$$

The pair (x, y) , — the so-called *line element*, — is the argument of the Finslerian objects.

To evaluate the curvature tensor R^i_k , we use the well-known formula

$$K^2 R^i_k := 2 \frac{\partial \bar{G}^i}{\partial x^k} - \frac{\partial \bar{G}^i}{\partial y^j} \frac{\partial \bar{G}^j}{\partial y^k} - y^j \frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} + 2 \bar{G}^j \frac{\partial^2 \bar{G}^i}{\partial y^k \partial y^j} \quad (\text{A.59})$$

(which is tantamount to the definition (3.8.7) on p. 66 of the book [5]; $\bar{G}^i = \frac{1}{2} \gamma^i_{nm} y^n y^m$, with the Finslerian Christoffel symbols γ^i_{nm}). The concomitant tensors

$$R^i_{km} := \frac{1}{3K} \left(\frac{\partial(K^2 R^i_k)}{\partial y^m} - \frac{\partial(K^2 R^i_m)}{\partial y^k} \right) \quad (\text{A.60})$$

and

$$R_n{}^i{}_{km} := \frac{\partial(K R^i_{km})}{\partial y^n} \quad (\text{A.61})$$

arise.

The cyclic identity

$$R_j{}^i{}_{kl|t} + R_j{}^i{}_{lt|k} + R_j{}^i{}_{tk|l} = P_j{}^i{}_{ku} R^u{}_{lt} + P_j{}^i{}_{lu} R^u{}_{tk} + P_j{}^i{}_{tu} R^u{}_{kl} \quad (\text{A.62})$$

(the formula (3.5.3) on p. 58 of the book [5]) is valid in any Finsler space. If we contract the identity by $g^{jl}\delta^k_i$, we get:

$$g^{jl}\left(R_j^i{}_{il|t} + R_j^i{}_{lt|i} + R_j^i{}_{ti|l}\right) = P^{li}{}_{iu}R^u{}_{lt} + P^{li}{}_{lu}R^u{}_{ti} + P^{li}{}_{tu}R^u{}_{il}. \quad (\text{A.63})$$

Under the Landsberg condition, the tensor P_{ijkl} is symmetric in all of its four indices and, therefore, the identity (A.63) reduces to merely

$$g^{jl}\left(R_j^i{}_{il|t} + R_j^i{}_{lt|i} + R_j^i{}_{ti|l}\right) = 0 \quad (\text{A.64})$$

($|i$ stands for the Finslerian covariant derivative), which can be formulated as follows.

Proposition A1. *In any Landsberg case of a Finsler space, the tensor*

$$\rho_{ij} := \frac{1}{2}(R_i^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm} \quad (\text{A.65})$$

is covariantly conserved:

$$\rho^i{}_{j|i} \equiv 0. \quad (\text{A.66})$$

It is obvious that the tensor ρ_{ij} , as well as the curvature tensor $R_n^i{}_{km}$, is positively homogeneous of the degree zero with respect to the vectors y :

$$\rho_{ij}(x, \lambda y) = \rho_{ij}(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y. \quad (\text{A.67})$$

The explicit representation of the curvature tensor $R_n^i{}_{km}$ of the Finsleroid-Landsberg-case space \mathcal{FF}_g^{PD} will be evaluated in detail in Appendix B.

On the basis of the above coefficients (A.56)–(A.58), the Finslerian *connection coefficients* $\Gamma^k{}_{ij} = \Gamma^k{}_{ij}(x, y)$ are constructed according to the well-known conventional rule:

$$\Gamma^k{}_{ij} = \gamma^k{}_{ij} - \bar{G}^n{}_i C_n^k{}_j - \bar{G}^m{}_j C_n^k{}_i + \bar{G}^{kn} C_{nij} \quad (\text{A.68})$$

with

$$\bar{G}^n{}_i = \gamma^n{}_{ij}y^j - 2\bar{G}^m C_m^{}{}^n{}_i = \Gamma^n{}_{ij}y^j = \frac{1}{2}G^n{}_i \quad (\text{A.69})$$

and

$$2\bar{G}^m = \gamma^m{}_{ij}y^i y^j = \bar{G}^m{}_i y^i = \Gamma^m{}_{ij}y^i y^j = G^m, \quad (\text{A.70})$$

where $C_{nij} = A_{nij}K^{-1}$ and $C_n^k{}_j = A_n^k{}_j K^{-1}$. By the help of these coefficients the *h-covariant derivatives* of tensors are constructed as exemplified by

$$\rho_{i|j} := \partial_j \rho_i - \bar{G}^k{}_j \frac{\partial \rho_i}{\partial y^k} - \Gamma^k{}_{ij} \rho_k, \quad (\text{A.71})$$

$$\rho^i{}_{|j} := \partial_j \rho^i - \bar{G}^k{}_j \frac{\partial \rho^i}{\partial y^k} + \Gamma^i{}_{kj} \rho^k, \quad (\text{A.72})$$

$$\rho^i{}_{k|j} := \partial_j \rho^i{}_k - \bar{G}^m{}_j \frac{\partial \rho^i{}_k}{\partial y^m} + \Gamma^i{}_{nj} \rho^n{}_k - \Gamma^h{}_{kj} \rho^i{}_h, \quad (\text{A.73})$$

and

$$g_{mn|j} = \partial_j g_{mn} - \bar{G}^h_j \frac{\partial g_{mn}}{\partial y^h} - \Gamma^h_{mj} g_{hn} - \Gamma^h_{nj} g_{mh}, \quad (\text{A.74})$$

where again $\partial_j = \partial/\partial x^j$. The coefficients (A.68) are *symmetric*:

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \quad (\text{A.75})$$

The importance of the covariant derivative $|j$ thus introduced is the property that the derivative is *metric*, that is we have the following identical vanishing:

$$K_{|j} = 0, \quad y^i_{|j} = 0, \quad l^i_{|j} = 0, \quad (\text{A.76})$$

and

$$g_{mn|j} = 0. \quad (\text{A.77})$$

The inverse implication can readily be made, namely the conditions (A.75)–(A.77) entail uniquely the coefficients (A.68)–(A.70). We apply the above covariant derivative $|i$ in the Finsleroid approach developed. The derivative $|i$ extends the Riemannian covariant derivative, such that

$$(|i)_{|g=0} = \nabla_i \quad \text{and} \quad (\Gamma^k_{ij})_{|g=0} = a^k_{ij}. \quad (\text{A.78})$$

If we apply the derivative definition (A.71) to the vector field $b_i(x)$, and the definition (A.72) to the vector field $b^i(x)$, then, because the fields are independent of y , we obtain merely

$$b_{i|j} := \partial_j b_i - \Gamma^m_{ij} b_m \quad (\text{A.79})$$

and

$$b^i_{|j} := \partial_j b^i + \Gamma^i_{mj} b^m. \quad (\text{A.80})$$

Therefore,

$$b_{|j} = y^i \left(\partial_j b_i - \Gamma^m_{ij} b_m \right) = \partial_j b - \bar{G}^m_j b_m$$

(the formula (A.69) has been used), or

$$b_{|j} = \nabla_j b - \left(\bar{G}^m_j - a^m_{jk} y^k \right) b_m. \quad (\text{A.81})$$

If we apply the formulas (B.18) and (B.6) of Appendix B, we just conclude that the last term in the right-hand part of the last equality (A.81) vanishes, so that the following proposition is valid.

Proposition A2. *In any Landsberg case of the \mathcal{FF}_g^{PD} -space, the covariant derivatives of the 1-form $b = b_i(x)y^i$ with respect to the Finsler connection and with respect to the associated Riemannian connection are equal to one another.*

$$b_{|j} = \nabla_j b. \quad (\text{A.82})$$

Since

$$b_{|j} = y^n \nabla_j b_n = k v_j$$

(see (1.8)), we have

$$b_{|j} = k v_j. \quad (\text{A.83})$$

Considering the associated Riemannian metric function

$$S = \sqrt{a_{mn}y^m y^n}, \quad (\text{A.84})$$

we have $(S^2)_{|j} = -2gq\nabla_j b$ (use (B.20)), that is,

$$(S^2)_{|j} = -2kqgv_j. \quad (\text{A.85})$$

Thus, the following proposition is valid.

Proposition A3. *In any Landsberg case of the \mathcal{FF}_g^{PD} -space, the covariant derivatives of the square of the associated Riemannian metric function with respect to the Finsler connection is given by (A.85).*

Taking into account

$$S^2 = q^2 + b^2, \quad (\text{A.86})$$

from (A.83) and (A.85) we may conclude that

$$q_{|k} = -\frac{1}{q}(b + gq)\nabla_k b. \quad (\text{A.87})$$

Next, we get

$$(bq)_{|k} = \frac{1}{q}[q^2 - b(b + gq)]\nabla_k b, \quad (q/b)_{|k} = -\frac{B}{qb^2}\nabla_k b, \quad (b/q)_{|k} = \frac{B}{q^3}\nabla_k b, \quad (\text{A.88})$$

and

$$J_{|n} = \frac{g}{2q}J\nabla_n b, \quad B_{|n} = -\frac{gB}{q}\nabla_n b. \quad (\text{A.89})$$

This leads to

$$Kb_{|k}l^k = kq^2, \quad \lambda_{|k}l^k = \frac{B}{q^3}b_{|k}l^k \quad (\text{A.90})$$

(where $\lambda = b/q$), and

$$SS_{|k}l^k = -gqb_{|k}l^k, \quad q_{|k}l^k = -\frac{1}{q}(b + gq)b_{|k}l^k, \quad (\text{A.91})$$

$$(bq)_{|k}l^k = \frac{1}{q}[q^2 - b(b + gq)]b_{|k}l^k, \quad B_{|k}l^k = -\frac{gB}{q}b_{|k}l^k, \quad (\text{A.92})$$

where $l^k = y^k/K$ is the unit tangent vector.

From (A.80) it ensues that

$$\left(b_{|j}^i\right)_{|y^h=b^h} = \nabla_j b^i + \Delta^i_{mj}b^m \quad (\text{A.93})$$

with the *osculating deflection coefficients*

$$\Delta^i_{mj} = \Delta^i_{mj}(x) = \Gamma^i_{mj}(x, b(x)) - a^i_{mj}(x), \quad (\text{A.94})$$

where $a^i_{mj}(x)$ are the Riemannian Christoffel symbols given by (A.43).

Let us take the derivative (A.72) over the field $y^h = b^h(x)$. Introducing the *osculating field*

$$\rho_{\{b\}}^i = \rho_{\{b\}}^i(x) = \rho_{|_{y^h=b^h}}^i \equiv \rho^i(x, b(x)), \quad (\text{A.95})$$

from (A.72), (A.93), and (A.95) it follows that

$$\left(\rho_{|j}^i\right)_{|_{y^h=b^h}} = \nabla_j \rho_{\{b\}}^i - \left(\nabla_j b^k + \Delta^k_{mj} b^m\right) \left(\frac{\partial \rho^i}{\partial y^k}\right)_{|_{y^h=b^h}} + \Delta^i_{kj} \rho_{\{b\}}^k. \quad (\text{A.96})$$

In the Landsberg case of the space \mathcal{FF}_g^{PD} under consideration, we have

$$\nabla_j b^k = k(\delta_j^k - b^k b_j) \quad (\text{A.97})$$

(see (1.8)) and

$$\Delta^i_{kj} b^k = 0 \quad (\text{A.98})$$

(see (B.6) and (B.20) in Appendix B). If also ρ^i is positively homogeneous of the degree one with respect to the vectors y :

$$\rho^i(x, \lambda y) = \lambda \rho^i(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y, \quad (\text{A.99})$$

then the identity

$$y^k \frac{\partial \rho^i}{\partial y^k} = \rho^i \quad (\text{A.100})$$

holds, entailing

$$b^k \left(\frac{\partial \rho^i}{\partial y^k}\right)_{|_{y^h=b^h}} = \rho_{\{b\}}^i. \quad (\text{A.101})$$

Under these conditions, the representation (A.96) reduces to the equality

$$\left(\rho_{|j}^i\right)_{|_{y^h=b^h}} = \nabla_j \rho_{\{b\}}^i - k \left[\left(\frac{\partial \rho^i}{\partial y^j}\right)_{|_{y^h=b^h}} - b_j \rho_{\{b\}}^i \right] + \Delta^i_{kj} \rho_{\{b\}}^k, \quad (\text{A.102})$$

which can also be written as

$$\left(\rho_{|j}^i\right)_{|_{y^h=b^h}} = \nabla_j \rho_{\{b\}}^i - k \left(\frac{\partial(\rho^i/b)}{\partial y^j} \right)_{|_{y^h=b^h}} + \Delta^i_{kj} \rho_{\{b\}}^k. \quad (\text{A.103})$$

Contracting here the indices yields

$$\left(\rho_{|i}^i\right)_{|_{y^h=b^h}} = \nabla_i \rho_{\{b\}}^i - k \left(\frac{\partial(\rho^i/b)}{\partial y^i} \right)_{|_{y^h=b^h}} + \Delta^i_{ki} \rho_{\{b\}}^k. \quad (\text{A.104})$$

In the particular case when the field $\rho_{\{b\}}^i$ is a factor of b^i , that is when

$$\rho_{\{b\}}^i = \gamma_{\{b\}} b^i, \quad \gamma_{\{b\}} = \gamma_{\{b\}}(x), \quad (\text{A.105})$$

the vanishing (A.98) suppresses the last term in (A.102), giving

$$\left(\rho_{|i}^i\right)_{|_{y^h=b^h}} = \nabla_i \rho_{\{b\}}^i - k \left(\frac{\partial(\rho^i/b)}{\partial y^i} \right)_{|_{y^h=b^h}}. \quad (\text{A.106})$$

Thus we have

Proposition A4. *With the conditions (A.97)–(A.105) fulfilled, the Finsler-covariant conservation law*

$$\rho_{|i}^i = 0 \quad (\text{A.107})$$

entails the Riemannian-divergence law

$$\nabla_i \rho_{\{b\}}^i = k \left(\frac{\partial(\rho^i/b)}{\partial y^i} \right) \Big|_{y^h=b^h} \quad (\text{A.108})$$

osculating along the fundamental vector field $b^i(x)$.

Notice that with (A.97) and (A.105) we can obtain the equality

$$\nabla_i \rho_{\{b\}}^i = b^i \partial_i \gamma_{\{b\}} + (N-1)k \gamma_{\{b\}}. \quad (\text{A.109})$$

Similar procedure of projecting on the vector field $b^i(x)$ can be addressed to the conservation law of the form $\rho^i_{j|i} = 0$.

In terms of the b -geodesic coordinates $\{z^A\}$, evaluation of the components of the Christoffel symbols (A.43) on the basis of the warped metric function (1.14) yields the following list:

$$s^N_{NN} = 0, \quad s^a_{NN} = 0, \quad s^N_{bN} = 0, \quad (\text{A.110})$$

$$s^a_{bN} = \frac{1}{2} r^{ac} \frac{\partial r_{bc}}{\partial z^N}, \quad s^N_{bc} = -\frac{1}{2} \frac{\partial r_{bc}}{\partial z^N}, \quad s^a_{bc} = \frac{1}{2} r^{ad} \left(\frac{\partial r_{dc}}{\partial z^b} + \frac{\partial r_{bd}}{\partial z^c} - \frac{\partial r_{bc}}{\partial z^d} \right). \quad (\text{A.111})$$

The equalities $b_N = 1$ and $b_a = 0$ entail

$$(\nabla b)_{NN} = 0, \quad (\text{A.112})$$

$$(\nabla b)_{Na} = 0, \quad (\text{A.113})$$

$$(\nabla b)_{aN} = 0, \quad (\text{A.114})$$

and

$$(\nabla b)_{ab} = \frac{1}{2} \frac{\partial r_{ab}}{\partial z^N}. \quad (\text{A.115})$$

When we apply the factorization

$$r_{ab}(z^A) = (\phi(z^A))^2 p_{ab}(z^c), \quad (\text{A.116})$$

so that

$$(ds)^2 = (dz^N)^2 + (\phi(z^A))^2 p_{ab}(z^c) dz^a dz^b, \quad (\text{A.117})$$

we obtain

$$\frac{\partial \phi}{\partial z^N} = k \phi \quad (\text{A.118})$$

and (A.115) reduces to

$$(\nabla b)_{ab} = k r_{ab}. \quad (\text{A.119})$$

The positive-definite \mathcal{FF}_g^{PD} -space described possesses the indefinite (relativistic) version, to be denoted as the \mathcal{FF}_g^{SR} -space (with the upperscripts “SR” meaning “special-relativistic”). The transition from the first space to the second space implies the formal change

$$g \rightarrow ig$$

of the Finsleroid parameter g , where i stands for the imaginary unity. The underlined space $\mathcal{R}_N = \{M, a_{mn}\}$ is now taken to be *pseudo-Riemannian*, such that the input metric tensor $\{a_{mn}(x)\}$ is to be pseudo-Riemannian with the *time-space signature*:

$$\text{sign}(a_{mn}) = (+ - - \dots). \quad (\text{A.120})$$

Generalizing accordingly the pseudo-Riemannian geometry in a pseudo-Finsleroid Finslerian way, we are to adapt the consideration to the following decomposition of the tangent bundle TM :

$$TM = \mathcal{S}_g^+ \cup \Sigma_g^+ \cup \mathcal{R}_g \cup \Sigma_g^- \cup \mathcal{S}_g^-, \quad (\text{A.121})$$

which sectors relate to the cases that the tangent vectors $y \in TM$ are, respectively, time-like, upper-cone isotropic, space-like, lower-cone isotropic, or past-like. The sectors are defined according to the following list:

$$\mathcal{S}_g^+ = \left(y \in \mathcal{S}_g^+ : y \in T_x M, b(x, y) > -g_-(x)q(x, y) \right), \quad (\text{A.122})$$

$$\Sigma_g^+ = \left(y \in \Sigma_g^+ : y \in T_x M, b(x, y) = -g_-(x)q(x, y) \right), \quad (\text{A.123})$$

$$\mathcal{R}_g^+ = \left(y \in \mathcal{R}_g^+ : y \in T_x M, -g_-(x)q(x, y) > b(x, y) > 0 \right), \quad (\text{A.124})$$

$$\mathcal{R}^0 = \left(y \in \mathcal{R}^0 : y \in T_x M, b(x, y) = 0 \right), \quad (\text{A.125})$$

$$\mathcal{R}_g^- = \left(y \in \mathcal{R}_g^- : y \in T_x M, 0 > b(x, y) > -g_+(x)q(x, y) \right), \quad (\text{A.126})$$

$$\Sigma_g^- = \left(y \in \Sigma_g^- : y \in T_x M, b(x, y) = -g_+(x)q(x, y) \right), \quad (\text{A.127})$$

$$\mathcal{S}_g^- = \left(y \in \mathcal{S}_g^- : y \in T_x M, b(x, y) < -g_+(x)q(x, y) \right), \quad (\text{A.128})$$

$$\mathcal{R}_g = \mathcal{R}_g^+ \cup \mathcal{R}_g^- \cup \mathcal{R}^0. \quad (\text{A.129})$$

We use the convenient notation

$$G = \frac{g}{h}, \quad h = \sqrt{1 + \frac{1}{4}g^2} \quad (\text{A.130})$$

(instead of (A.2)),

$$g_+ = -\frac{1}{2}g + h, \quad g_- = -\frac{1}{2}g - h, \quad (\text{A.131})$$

$$G_+ = \frac{g_+}{h} \equiv -\frac{1}{2}G + 1, \quad G_- = \frac{g_-}{h} \equiv -\frac{1}{2}G - 1, \quad (\text{A.132})$$

$$g^+ = \frac{1}{g_+} = -g_-, \quad g^- = \frac{1}{g_-} = -g_+, \quad (\text{A.133})$$

$$g^+ = \frac{1}{2}g + h, \quad g^- = \frac{1}{2}g - h, \quad (\text{A.134})$$

$$G^+ = \frac{g^+}{h} \equiv \frac{1}{2}G + 1, \quad G^- = \frac{g^-}{h} \equiv \frac{1}{2}G - 1. \quad (\text{A.135})$$

The following identities hold

$$g_+ + g_- = -g, \quad g_+ - g_- = 2h, \quad (\text{A.136})$$

$$g^+ + g^- = g, \quad g^+ - g^- = 2h, \quad (\text{A.137})$$

$$g_+ g_- = -1, \quad g^+ g^- = -1, \quad (\text{A.138})$$

together with the g -symmetry

$$g_+ \xrightarrow{g \rightarrow -g} -g_-, \quad g^+ \xrightarrow{g \rightarrow -g} -g^-, \quad G_+ \xrightarrow{g \rightarrow -g} -G_-, \quad G^+ \xrightarrow{g \rightarrow -g} -G^-. \quad (\text{A.139})$$

It is implied that $g = g(x)$ is a scalar on the underlying manifold M . All the range

$$-\infty < g(x) < \infty \quad (\text{A.140})$$

(instead of (A.1)) is now admissible. We also assume that the manifold M admits a 1-form $b = b(x, y)$ which is *timelike* in terms of the pseudo-Riemannian metric \mathcal{S} , such that the pseudo-Riemannian length of the involved vector b_i be equal to 1. With respect to natural local coordinates in the space \mathcal{R}_N we have the local representations

$$b = b_i(x) y^i, \quad (\text{A.141})$$

$$S = \sqrt{|a_{ij}(x) y^i y^j|}, \quad (\text{A.142})$$

$$q = \sqrt{|r_{ij}(x) y^i y^j|}, \quad (\text{A.143})$$

$$r_{ij}(x) = b_i(x) b_j(x) - a_{ij}(x), \quad (\text{A.144})$$

$$a^{ij} b_i b_j = 1, \quad (\text{A.145})$$

$$S^2 = b^2 - q^2, \quad (\text{A.146})$$

$$b^i r_{ij} = 0, \quad (\text{A.147})$$

where

$$b^i := a^{ik} b_k \quad (\text{A.148})$$

(compare with (A.18)–(A.21)).

The *pseudo-Finsleroid characteristic quadratic form*

$$B(x, y) := b^2 - gqb - q^2 \equiv (b + g_+q)(b + g_-q) \quad (\text{A.149})$$

is now of the positive discriminant

$$D_{\{B\}} = 4h^2 > 0 \quad (\text{A.150})$$

(compare these formulas with (A.3) and (A.4)).

In terms of these concepts, we propose

Definition. The scalar function $F(x, y)$ given by the formula

$$F(x, y) := \sqrt{|B(x, y)|} J(x, y) \equiv |b + g_-q|^{G_+/2} |b + g_+q|^{-G_-/2}, \quad (\text{A.151})$$

where

$$J(x, y) = \left| \frac{b + g_-q}{b + g_+q} \right|^{-G/4}, \quad (\text{A.152})$$

is called the *pseudo-Finsleroid-Finsler metric function*.

Again, the zero-vector $y = 0$ is excluded from consideration:

$$y \neq 0. \quad (\text{A.153})$$

The normalization of the above function F is compatible with fulfilling the unit property

$$F(x, b^i(x)) = 1 \quad (\text{A.154})$$

(similarly to (A.11)).

The positive (not absolute) homogeneity holds:

$$F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0, \quad \forall x, \quad \forall y. \quad (\text{A.155})$$

The functions

$$L(x, y) = q - \frac{g}{2}b \quad (\text{A.156})$$

and

$$A(x, y) = b - \frac{g}{2}q \quad (\text{A.157})$$

are now to be used instead of (A.10) and (A.12), so that (A.13) changes to read

$$L^2 - h^2 b^2 = B, \quad A^2 - h^2 q^2 = B. \quad (\text{A.158})$$

Similarly to (A.14), we introduce

Definition. The arisen space

$$\mathcal{FF}_g^{SR} := \{\mathcal{R}_N; b(x, y); g(x); F(x, y)\} \quad (\text{A.159})$$

is called the *pseudo-Finsleroid-Finsler space*.

The upperscript “SR” emphasizes the Specially Relativistic character of the space under study.

Definition. The space $\mathcal{R}_N = (M, \mathcal{S})$ entering the above definition is called the *associated pseudo-Riemannian space*.

Definition. The scalar $g(x)$ is called the *pseudo-Finsleroid charge*. The 1-form b is called the *pseudo-Finsleroid-axis 1-form*.

It can be verified that the Finslerian metric tensor constructed from the function F given by (A.151) does inherit from the tensor $\{a_{ij}(x)\}$ the time-space signature (A.120):

$$\text{sign}(g_{ij}) = (+ - - \dots). \quad (\text{A.160})$$

The structure (A.42) for the Cartan tensor remains valid in the pseudo-Finsleroid case, now with

$$A_h A^h = -\frac{N^2}{4} g^2 \quad (\text{A.161})$$

(compare with (A.40)). Elucidating the structure of the respective indicatrix curvature tensor (2.29) of the \mathcal{FF}_g^{SR} -space again results in the conclusion that the indicatrix curvature value $\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}}$ is

$$\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} = -\left(1 + \frac{1}{4}g^2\right) \leq -1, \quad (\text{A.162})$$

so that

$$\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} \xrightarrow{g \rightarrow 0} \mathcal{R}_{\text{pseudo-Euclidean Sphere}} = -1.$$

The pseudo-Finsleroid indicatrix is a space of constant negative curvature.

Again, it is convenient to use the variables

$$v^i := y^i - b b^i, \quad v_m := u_m - b b_m = r_{mn} y^n \equiv r_{mn} v^n \equiv a_{mn} v^n, \quad (\text{A.163})$$

where

$$r_{mn} = a_{mn} - b_m b_n \quad (\text{A.164})$$

is the projection tensor. We obtain the identities

$$r^i_n := a^{im} r_{mn} = \delta^i_n - b^i b_n = \frac{\partial v^i}{\partial y^n}, \quad (\text{A.165})$$

and

$$v_i b^i = v^i b_i = 0, \quad r_{ij} b^j = r^i_j b^j = b_i r^i_j = 0. \quad (\text{A.166})$$

It is useful to apply the notation

$$q = \sqrt{r_{mn} v^m v^n}. \quad (\text{A.167})$$

From the pseudo-Finsleroid-Finsler metric function (A.151) we straightforwardly evaluate the explicit representations

$$y_i = \left(v_i + (b - gq) b_i \right) \frac{F^2}{B} \quad (\text{A.168})$$

and

$$g_{ij} = \left[a_{ij} + \frac{g}{B} \left(-q(b + gq) b_i b_j - q(b_i v_j + b_j v_i) - b \frac{v_i v_j}{q} \right) \right] \frac{F^2}{B}. \quad (\text{A.169})$$

Through the reciprocity $(g^{ij}) = (g_{ij})^{-1}$, we arrive at the representations

$$g^{ij} = \left[a^{ij} + \frac{g}{q}(bb^ib^j - b^iy^j - b^jy^i) + \frac{g}{Bq}(b - gq)y^iy^j \right] \frac{B}{F^2} \quad (\text{A.170})$$

and

$$g^{ij} = \left[a^{ij} + \frac{g}{B} \left(bq b^ib^j + q(b^iv^j + b^jv^i) + (b - gq)\frac{v^iv^j}{q} \right) \right] \frac{B}{F^2}. \quad (\text{A.171})$$

It is obvious that

$$g_{ij}(x, y)|_{q=0} = a_{ij}(x) \quad (\text{A.172})$$

and

$$g_{ij}(x, b^n(x)) = a_{ij}(x). \quad (\text{A.173})$$

We observe the phenomenon that the representations (A.168)–(A.171) are directly obtainable from the positive-definite case representations (A.23)–(A.26) through the formal change:

$$g \xrightarrow{PD \rightarrow SR} ig \quad (\text{A.174})$$

and

$$q \xrightarrow{PD \rightarrow SR} iq, \quad (\text{A.175})$$

where i stands for the imaginary unity. Therefore, we may apply the rules

$$\frac{g}{q} \xrightarrow{PD \rightarrow SR} \frac{g}{q}, \quad gq \xrightarrow{PD \rightarrow SR} -gq. \quad (\text{A.176})$$

It is the useful exercise to verify that if we apply these rules to the expression (A.151) of the relativistic function F , we obtain the positive-definite case function K defined by (A.6).

Appendix B: Evaluation and application of curvature tensor

In addition to the variables

$$v^m = y^m - bb^m, \quad v_n = a_{mn}v^m \quad (\text{B.1})$$

(see (A.18)), we shall use the tensor

$$v^m{}_n := \frac{\partial v^m}{\partial y^n} + \frac{1}{q^2}v^mv_n,$$

so that,

$$v^m{}_n = r_n^m + \frac{1}{q^2}v^mv_n \quad (\text{B.2})$$

$$b^nv^m{}_n = 0, \quad (\text{B.3})$$

and

$$y^nv^m{}_n = 2v^m, \quad (\text{B.4})$$

together with the tensor

$$v^m_{hn} := v^m_h v_n + v^m_n v_h + v^m v_{hn} - \frac{4}{q^2} v^m v_h v_n \quad (\text{B.5})$$

which fulfills the identities

$$b^n v^m_{hn} = b_m v^m_{hn} = 0, \quad y^n v^m_{hn} = q^2 v^m_h. \quad (\text{B.6})$$

We shall also use the notation

$$s_i = y^j \nabla_i b_j, \quad s^i = a^{in} s_n, \quad (ys) = y^i s_i. \quad (\text{B.7})$$

These objects don't depend on the vectors y .

Below, we shall evaluate the curvature tensor of the \mathcal{FF}_g^{PD} -space in the Landsberg case, that is, when the conditions

$$g = \text{const} \quad (\text{B.8})$$

and

$$\nabla_i b_n = k r_{in} \quad (\text{B.9})$$

(see (1.8) and (1.23)) are satisfied. The formulae indicated in (B.7) reduce to

$$s_i = k v_i, \quad s^i = k v^i, \quad (ys) = k q^2. \quad (\text{B.10})$$

We shall denote

$$k_i = \frac{\partial k}{\partial x^i} \quad (\text{B.11})$$

and

$$(yk) = y^i k_i, \quad (\text{B.12})$$

and also

$$\tilde{k}_i = k_i + k^2 b_i \quad (\text{B.13})$$

together with

$$(y\tilde{k}) = y^i \tilde{k}_i = (yk) + k^2 b \quad (\text{B.14})$$

and

$$n_1 = (y\tilde{k}) - b(b\tilde{k}). \quad (\text{B.15})$$

The spray coefficients G^m entailed have been found in the previous work [11-12]. They read merely

$$G^m = k g q v^m + a^m_{jn} y^j y^n. \quad (\text{B.16})$$

It is notable that the Finsleroid-Finsler metric function K does not enter the right-hand side of these G^m . The presence of the constant g in the right-hand side of (B.16) is the only trace of the function K in these coefficients.

The coefficients

$$G^m_n = \frac{\partial G^m}{\partial y^n} \quad (\text{B.17})$$

equal

$$G^m_n = k g q v^m_n + 2 a^m_{jn} y^j, \quad (\text{B.18})$$

and the coefficients

$$G^m_{nh} = \frac{\partial G^m_n}{\partial y^h} \quad (\text{B.19})$$

equal

$$G^m{}_{nh} = \frac{kg}{q}v^m{}_{hn} + 2a^m{}_{hn}, \quad (\text{B.20})$$

where the tensor (B.5) has been used.

Differentiating (B.16) yields

$$\frac{\partial G^m}{\partial x^i} = gqk_i v^m - kgq(s_i b^m + b\nabla_i b^m) - kg\frac{1}{w}s_i v^m + \Delta. \quad (\text{B.21})$$

Here and below, $w = q/b$ and Δ symbolizes the summary of the terms which involve partial derivatives of the input Riemannian metric tensor a_{ij} with respect to the coordinate variables x^k .

Simple direct calculations yield

$$\begin{aligned} \frac{\partial^2 G^m}{\partial y^k \partial x^i} &= g\frac{1}{q}k_i v_k v^m + gqk_i r^m{}_k - kg\frac{1}{q}v_k(s_i b^m + b\nabla_i b^m) - kgq(b^m \nabla_k b_i + b_k \nabla_i b^m) \\ &\quad - kg\frac{1}{q}b_k v^m s_i + kg\frac{b}{q^3}v_k v^m s_i - kg\frac{1}{w}v^m \nabla_k b_i - kg\frac{1}{w}s_i r^m{}_k + \Delta \end{aligned} \quad (\text{B.22})$$

and

$$\begin{aligned} y^i \frac{\partial^2 G^m}{\partial y^k \partial x^i} &= g\frac{1}{q}(yk)v_k v^m + gq(yk)r^m{}_k - kg\frac{1}{q}v_k((ys)b^m + bs^m) - kgq(b^m s_k + b_k s^m) \\ &\quad - kg\frac{1}{q}b_k v^m(ys) + kg\frac{b}{q^3}v_k v^m(ys) - kg\frac{1}{w}v^m s_k - kg\frac{1}{w}(ys)r^m{}_k + \Delta, \end{aligned} \quad (\text{B.23})$$

together with

$$\begin{aligned} 2\frac{\partial \bar{G}^i}{\partial x^k} - y^j \frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} &= gqk_k v^i - kgq(s_k b^i + b\nabla_k b^i) - kg\frac{1}{w}s_k v^i \\ &\quad - \frac{1}{2}g\frac{1}{q}(yk)v_k v^i - \frac{1}{2}gq(yk)r^i{}_k + \frac{1}{2}kg\frac{1}{q}v_k((ys)b^i + bs^i) + \frac{1}{2}kgq(b^i s_k + b_k s^i) \\ &\quad + \frac{1}{2}kg\frac{1}{q}b_k v^i(ys) - \frac{1}{2}kg\frac{b}{q^3}v_k v^i(ys) + \frac{1}{2}kg\frac{1}{w}v^i s_k + \frac{1}{2}kg\frac{1}{w}(ys)r^i{}_k + \Delta. \end{aligned} \quad (\text{B.24})$$

This entails

$$\begin{aligned} 2\frac{\partial \bar{G}^i}{\partial x^k} - y^j \frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} &= -\frac{1}{2}gq(yk)\left(r^i{}_k + \frac{1}{q^2}v^i v_k\right) + gqk_k v^i \\ &\quad - \frac{1}{2}kgqs_k b^i - kgqb\nabla_k b^i - \frac{1}{2}kg\frac{1}{w}s_k v^i + \frac{1}{2}kg\frac{1}{q}v_k((ys)b^i + bs^i) + \frac{1}{2}kgqb_k s^i \\ &\quad + \frac{1}{2}kg\frac{1}{q}b_k v^i(ys) + \frac{1}{2}kg\frac{1}{w}(ys)\left(r^i{}_k - \frac{1}{q^2}v^i v_k\right) + \Delta \end{aligned} \quad (\text{B.25})$$

and

$$G^i_j G^j_k = k^2 g^2 q^2 \left(v^i_k + \frac{2}{q^2} v^i v_k \right) + \Delta, \quad (\text{B.26})$$

together with

$$G^j \frac{\partial^2 G^i}{\partial y^k \partial y^j} = k^2 g^2 q^2 v^i_k + \Delta \quad (\text{B.27})$$

and

$$-\frac{\partial \bar{G}^i}{\partial y^j} \frac{\partial \bar{G}^j}{\partial y^k} + 2 \bar{G}^j \frac{\partial^2 \bar{G}^i}{\partial y^k \partial y^j} = \frac{1}{4} k^2 g^2 q^2 \left(v^i_k - \frac{2}{q^2} v^i v_k \right) + \Delta. \quad (\text{B.28})$$

With these formulae, we can obtain the explicit form of the curvature tensor R^i_k on the basis of the definition (A.24) written in the previous Appendix A. The result reads

$$\begin{aligned} K^2 R^i_k &= -\frac{1}{2} g q (y k) \left(r^i_k + \frac{1}{q^2} v^i v_k \right) + g q k_k v^i \\ &\quad - \frac{1}{2} k g q s_k b^i - k g q b \nabla_k b^i - \frac{1}{2} k g \frac{1}{w} s_k v^i + \frac{1}{2} k g \frac{1}{q} v_k \left((y s) b^i + b s^i \right) + \frac{1}{2} k g q b_k s^i \\ &\quad + \frac{1}{2} k g \frac{1}{q} b_k v^i (y s) + \frac{1}{2} k g \frac{1}{w} (y s) \left(r^i_k - \frac{1}{q^2} v^i v_k \right) \\ &\quad + \frac{1}{4} k^2 g^2 q^2 \left(r^i_k - \frac{1}{q^2} v^i v_k \right) + \Delta. \end{aligned} \quad (\text{B.29})$$

Using the claimed equalities $\nabla_m b_n = k r_{mn}$, $s_i = k v_i$, $s^i = k v^i$, $(y s) = k q^2$ (see (B.9) and (B.10)) evokes much simplification, leaving us with

$$\begin{aligned} K^2 R^i_k &= -\frac{1}{2} g q (y k) \left(r^i_k + \frac{1}{q^2} v^i v_k \right) + g q k_k v^i \\ &\quad + k^2 g q b_k v^i - \frac{1}{2} k^2 g q b \left(r^i_k + \frac{1}{q^2} v^i v_k \right) + \frac{1}{4} k^2 g^2 q^2 \left(r^i_k - \frac{1}{q^2} v^i v_k \right) + \Delta. \end{aligned} \quad (\text{B.30})$$

Eventually,

$$K^2 R^i_k = \frac{1}{4} k^2 g^2 q^2 \left(r^i_k - \frac{1}{q^2} v^i v_k \right) - \frac{1}{2} g q (y \tilde{k}) \left(r^i_k + \frac{1}{q^2} v^i v_k \right) + g q \tilde{k}_k v^i + y^n a_n{}^i{}_{km} y^m. \quad (\text{B.31})$$

Thus we have arrived at the following proposition.

Proposition B1. *In the Finsleroid space \mathcal{FF}_g^{PD} , the Landsberg-case curvature tensor R^i_k is explicitly given by the previous formula.*

The entailed tensor (A.25) is found to read

$$3K R^i_{km} = \frac{3}{4} k^2 g^2 (v_m r^i_k - v_k r^i_m)$$

$$\begin{aligned}
& -\frac{1}{2}gq \left[\tilde{k}_m \left(r^i_k + \frac{1}{q^2} v^i v_k \right) - \tilde{k}_k \left(r^i_m + \frac{1}{q^2} v^i v_m \right) \right] \\
& + \frac{g}{q} (v_m \tilde{k}_k - v_k \tilde{k}_m) v^i + gq (\tilde{k}_k r^i_m - \tilde{k}_m r^i_k) + 3y^n a_n^i{}_{km} \\
& = \frac{3}{4} k^2 g^2 (v_m r^i_k - v_k r^i_m) \\
& - \frac{3}{2} gq \left[\tilde{k}_m \left(r^i_k + \frac{1}{q^2} v^i v_k \right) - \tilde{k}_k \left(r^i_m + \frac{1}{q^2} v^i v_m \right) \right] + 3y^n a_n^i{}_{km}.
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
KR^i{}_{km} &= \frac{1}{4} k^2 g^2 (v_m r^i_k - v_k r^i_m) \\
& - \frac{1}{2} gq \left[\tilde{k}_m \left(r^i_k + \frac{1}{q^2} v^i v_k \right) - \tilde{k}_k \left(r^i_m + \frac{1}{q^2} v^i v_m \right) \right] + y^n a_n^i{}_{km}. \tag{B.32}
\end{aligned}$$

The full curvature tensor (A.26) is obtained to read

$$\begin{aligned}
R_n^i{}_{km} &= \frac{1}{4} k^2 g^2 (r_{mn} r^i_k - r_{kn} r^i_m) \\
& - \frac{g}{2q} v_n \left[\tilde{k}_m \left(r^i_k - \frac{1}{q^2} v^i v_k \right) - \tilde{k}_k \left(r^i_m - \frac{1}{q^2} v^i v_m \right) \right] \\
& + \frac{g}{2q} \left[r^i_n (\tilde{k}_k v_m - \tilde{k}_m v_k) + v^i (\tilde{k}_k r_{mn} - \tilde{k}_m r_{kn}) \right] + a_n^i{}_{km}. \tag{B.33}
\end{aligned}$$

The tensor can also be represented in the short form

$$R_n^i{}_{km} = \frac{1}{4} k^2 g^2 (r_{mn} r^i_k - r_{kn} r^i_m) + \frac{g}{2q} (\tilde{k}_k r^i_{nm} - \tilde{k}_m r^i_{nk}) + a_n^i{}_{km}, \tag{B.34}$$

with the help of the tensor

$$r^i_{nm} = r_{nm} v^i + r^i_m v_n + r^i_n v_m - \frac{1}{q^2} v^i v_n v_m. \tag{B.35}$$

From (B.33) we obtain the contraction

$$R_n^i{}_{im} = \frac{1}{4} (N-2) k^2 g^2 r_{mn} - \frac{g}{2q} v_n \left[(N-2) \tilde{k}_m - \tilde{k}_i r^i_m + \frac{1}{q^2} \tilde{k}_i v^i v_m \right]$$

$$+\frac{g}{2q} \left[r^i_n \tilde{k}_i v_m - \tilde{k}_m v_n + v^i \tilde{k}_i r_{mn} - \tilde{k}_m v_n \right] + a_n^i{}_{im},$$

or, after due simplification,

$$R_n^i{}_{im} = \frac{1}{4}(N-2)k^2 g^2 r_{mn} - \frac{Ng}{2q} v_n \tilde{k}_m$$

$$+\frac{g}{2q}(v_n \tilde{k}_i r^i_m + v_m \tilde{k}_i r^i_n) + \frac{g}{2q} n_1 \left(r_{mn} - \frac{1}{q^2} v_m v_n \right) + a_n^i{}_{im}. \quad (\text{B.36})$$

Next, we find

$$a^{nm} R_n^i{}_{im} = \frac{1}{4}(N-2)(N-1)k^2 g^2 - \frac{Ng}{2q} n_1 + \frac{g}{q} n_1 + \frac{g}{2q}(N-2)n_1 + a^{nm} a_n^i{}_{im}. \quad (\text{B.37})$$

Let us now evaluate the scalar

$$R^{ni}{}_{in} = g^{nm} R_n^i{}_{im}, \quad (\text{B.38})$$

applying the representation

$$g^{nm} = \left[a^{nm} + \frac{g}{q}(bb^n b^m - b^n y^m - b^m y^n) + \frac{g}{Bq}(b + gq)y^n y^m \right] \frac{B}{K^2}$$

(see (A.25)). We obtain

$$\begin{aligned} \frac{K^2}{B} R^{ni}{}_{in} &= \frac{1}{4}(N-2)(N-1)k^2 g^2 + a^{nm} a_n^i{}_{im} - \frac{Ng}{2q} n_1 + \frac{g}{q} n_1 + \frac{g}{2q}(N-2)n_1 \\ &+ \frac{gb}{q} b^m b^n R_n^i{}_{im} - \frac{g}{q} y^m b^n R_n^i{}_{im} - \frac{g}{q} b^m y^n R_n^i{}_{im} + \frac{g}{Bq}(b + gq)y^m y^n R_n^i{}_{im} \\ &= \frac{1}{4}(N-2)(N-1)k^2 g^2 + \frac{g^2}{2} N b^m \tilde{k}_m \\ &\quad - \frac{Ng}{2q} n_1 + \frac{g}{q} n_1 + \frac{g}{2q}(N-2)n_1 \\ &+ \frac{g}{Bq}(b + gq) \left[\frac{1}{4}(N-2)k^2 g^2 q^2 - \frac{1}{2} gq \left(N(y\tilde{k}) - 2\tilde{k}_i v^i \right) \right] + \frac{K^2}{B} g^{nm} a_n^i{}_{im} \\ &= \frac{1}{4}(N-2)(N-1)k^2 g^2 + \frac{g^2}{2} N b^m \tilde{k}_m \\ &\quad - \frac{Ng}{2q} n_1 + \frac{g}{q} n_1 + \frac{g}{2q}(N-2)n_1 \end{aligned}$$

$$\begin{aligned}
& + \frac{g}{bq} \left[\frac{1}{4}(N-2)k^2g^2q^2 - \frac{1}{2}gq \left(N(y\tilde{k}) - 2\tilde{k}_i v^i \right) \right] \\
& - \frac{gq}{Bb} \left[\frac{1}{4}(N-2)k^2g^2q^2 - \frac{1}{2}gq \left(N(y\tilde{k}) - 2\tilde{k}_i v^i \right) \right] + \frac{K^2}{B} g^{nm} a_n^i{}_{im}.
\end{aligned}$$

Eventually, we obtain the representation

$$\begin{aligned}
\frac{K^2}{B} R^{ni}{}_{in} &= (N-2) \left[\frac{1}{4}(N-1)k^2g^2 + \frac{1}{4}\frac{1}{b}k^2g^3q - \frac{1}{2}\frac{1}{b}g^2n_1 \right] \\
& - \frac{Ng}{2q}n_1 + \frac{g}{q}n_1 + \frac{g}{2q}(N-2)n_1 \\
& - \frac{gq}{Bb} \left[\frac{1}{4}(N-2)k^2g^2q^2 - \frac{1}{2}gq \left(N(y\tilde{k}) - 2n_1 \right) \right] + \mu,
\end{aligned} \tag{B.39}$$

where

$$\mu := \frac{K^2}{B} g^{nm} a_n^i{}_{im}. \tag{B.40}$$

We have

$$\mu = \frac{K^2}{B} g^{nm} a_n^i{}_{im} = a^{nm} a_n^i{}_{im} + \frac{gb}{q} b^m b^n a_n^i{}_{im} - 2\frac{g}{q} y^m b^n a_n^i{}_{im} + \frac{g}{Bq} (b + gq) y^m y^n a_n^i{}_{im}. \tag{B.41}$$

Use the components

$$g_{ij} = \left[a_{ij} + \frac{g}{B} \left(q(b + gq) b_i b_j + q(b_i v_j + b_j v_i) - b \frac{v_i v_j}{q} \right) \right] \frac{K^2}{B}$$

(see (A.24)) to lower the index according to

$$R_{nikm} = g_{ji} R_n^j{}_{km}. \tag{B.42}$$

We obtain

$$\begin{aligned}
\frac{B}{K^2} R_{nikm} &= \frac{1}{4} k^2 g^2 (r_{mn} r_{ik} - r_{kn} r_{im}) \\
& - \frac{g}{2q} v_n \left[\tilde{k}_m \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) - \tilde{k}_k \left(r_{im} - \frac{1}{q^2} v_i v_m \right) \right] \\
& + \frac{g}{2q} \left[r_{in} (\tilde{k}_k v_m - \tilde{k}_m v_k) + v_i (\tilde{k}_k r_{mn} - \tilde{k}_m r_{kn}) \right] + \frac{B}{K^2} g_{ji} a_n^j{}_{km}
\end{aligned}$$

$$+ \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left[\frac{1}{4} k^2 g^2 (r_{mn} v_k - r_{kn} v_m) + \frac{g}{2q} \left(v_n (\tilde{k}_k v_m - \tilde{k}_m v_k) + q^2 (\tilde{k}_k r_{mn} - \tilde{k}_m r_{kn}) \right) \right], \quad (\text{B.43})$$

where

$$\frac{B}{K^2} g_{ji} a_n^j{}_{km} = a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + q v_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km}. \quad (\text{B.44})$$

Also, we must perform the involved calculation to find the contracted tensor

$$R^m{}_{ikm} = g^{nm} R_{nikm}. \quad (\text{B.45})$$

We obtain

$$\begin{aligned} R^m{}_{ikm} &= \frac{1}{4} k^2 g^2 (N-2) r_{ik} - \frac{g}{2q} v^m \tilde{k}_m \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\ &\quad + \frac{g}{2q} \left[(\tilde{k}_k v_i - r_i^m \tilde{k}_m v_k) + v_i \left((N-1) \tilde{k}_k - \tilde{k}_m r_k^m \right) \right] \\ &\quad + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left[\frac{1}{4} k^2 g^2 (N-2) v_k + \frac{g}{2q} \left((q^2 \tilde{k}_k - v^m \tilde{k}_m v_k) + q^2 \left((N-1) \tilde{k}_k - \tilde{k}_m r_k^m \right) \right) \right] \\ &\quad + \frac{g^2}{2} b^m \tilde{k}_m \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\ &\quad + \frac{g}{Bq} (b + gq) \left[\frac{1}{4} k^2 g^2 q^2 \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) - \frac{1}{2} gq (y \tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \right. \\ &\quad \left. + \frac{g}{q} v_i \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) gq \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) \right] + \theta_i, \end{aligned} \quad (\text{B.46})$$

where

$$\theta_i := g_{ji} a_n^j{}_{km} g^{nm}. \quad (\text{B.47})$$

Adding (B.37) and (B.46) yields

$$R^m{}_{ikm} + R_i{}^h{}_{hk} = \frac{1}{2} k^2 g^2 (N-2) r_{ik} + \frac{g^2}{2} b^m \tilde{k}_m \left(r_{ik} - \frac{1}{q^2} v_i v_k \right)$$

$$\begin{aligned}
& + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left[\frac{1}{4} k^2 g^2 (N-2) v_k + \frac{gq}{2} \left(\tilde{k}_m r_k^m - \frac{1}{q^2} \tilde{k}_m v^m v_k - N \tilde{k}_k \right) \right] \\
& + \frac{g}{Bq} (b + gq) \left[\frac{1}{4} k^2 g^2 q^2 \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) - \frac{1}{2} gq (y \tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \right. \\
& \quad \left. + \frac{g}{q} v_i \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) \right. \\
& \quad \left. + \frac{g^2 q^2}{B} b_i \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) - \frac{g^2 q}{B} b \frac{v_i}{q} \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) \right] + a_i^h{}_{hk} + \theta_i. \tag{B.48}
\end{aligned}$$

We find

$$\begin{aligned}
\theta_i &= a^{nm} \left[a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + qv_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km} \right] \\
&+ \frac{g}{q} b b^n b^m \left[a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + qv_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km} \right] \\
&- \frac{g}{q} b^n y^m \left[a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + qv_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km} \right] \\
&- \frac{g}{q} y^n b^m \left[a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + qv_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km} \right] \\
&+ \frac{g}{Bq} (b + gq) y^m y^n \left[a_{nikm} + \frac{g}{B} \left(q(b + gq) b_i + qv_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) v_j a_n^j{}_{km} \right].
\end{aligned}$$

The previous representation reduces to

$$\begin{aligned}
\theta_i &= a^{nm} \left[a_{nikm} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2) v_i \right) b_j a_n^j{}_{km} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) y^j a_{njkm} \right] \\
&+ \frac{g}{q} b b^n b^m \left[a_{nikm} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) y^j a_{njkm} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{g}{q}b^n y^m \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] \\
& -\frac{g}{q}y^n b^m \left[a_{nikm} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) b_j a_n^j{}_{km} \right] \\
& +\frac{g}{Bq}(b + gq)y^m y^n \left[a_{nikm} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) b_j a_n^j{}_{km} \right],
\end{aligned}$$

or

$$\begin{aligned}
\theta_i &= a^{nm} \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] + \frac{g}{Bq}(b + gq)y^m y^n a_{nikm} \\
& +\frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) b_j a_n^j{}_{km} a^{nm} \\
& +\frac{g}{q}bb^n b^m \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] \\
& -\frac{g}{q}b^n y^m \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] \\
& -\frac{g}{q}y^n b^m \left[a_{nikm} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) b_j a_n^j{}_{km} \right] \\
& +\frac{g}{Bq}(b + gq)\frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) b_j a_n^j{}_{km} y^m y^n. \tag{B.49}
\end{aligned}$$

Make required insertions

$$b_j a_n^j{}_{km} = \left(-\tilde{k}_k r_{nm} + \tilde{k}_m r_{nk} \right), \quad b^n a_n^h{}_{hm} = \tilde{k}_h r^h{}_m - (N-1)\tilde{k}_m \tag{B.50}$$

(see (A.48)) and obtain

$$\begin{aligned}
\theta_i &= a^{nm} \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] + \frac{g}{Bq}(b + gq)y^m y^n a_{nikm} \\
& -\frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) (N-1)\tilde{k}_k + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) r_k^m \tilde{k}_m \\
& +\frac{g}{q}bb^m \left[\tilde{k}_k r_{im} - \tilde{k}_m r_{ik} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j \left(\tilde{k}_k r_{jm} - \tilde{k}_m r_{jk} \right) \right] \\
& -\frac{g}{q}y^m \left[\tilde{k}_k r_{im} - \tilde{k}_m r_{ik} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j \left(\tilde{k}_k r_{jm} - \tilde{k}_m r_{jk} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{g}{q}y^n \left[-\tilde{k}_n r_{ki} + \tilde{k}_i r_{nk} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) \left(-\tilde{k}_k r_{nm} + \tilde{k}_m r_{nk} \right) b^m \right] \\
& - \frac{g}{Bq}(b + gq) \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) \left(q^2 \tilde{k}_k - (y\tilde{k})v_k \right). \tag{B.51}
\end{aligned}$$

We obtain the tensor

$$\begin{aligned}
R^m{}_{ikm} + R_i{}^h{}_{hk} &= \frac{1}{2}k^2 g^2 (N - 2)r_{ik} + \frac{g^2}{2}b^m \tilde{k}_m \left(r_{ik} - \frac{1}{q^2}v_i v_k \right) \\
&+ \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) \left[\frac{1}{4}k^2 g^2 (N - 2)v_k + \frac{gq}{2} \left(\tilde{k}_m r_k^m - \frac{1}{q^2} \tilde{k}_m v^m v_k - N\tilde{k}_k \right) \right] \\
&+ \frac{g}{Bq}(b + gq) \left[\frac{1}{4}k^2 g^2 q^2 \left(r_{ik} - \frac{1}{q^2}v_i v_k \right) - \frac{1}{2}gq(y\tilde{k}) \left(r_{ik} - \frac{1}{q^2}v_i v_k \right) \right. \\
&\quad \left. + \frac{g}{q}v_i \left(q^2 \tilde{k}_k - (y\tilde{k})v_k \right) \right] + a_i{}^h{}_{hk} \\
&+ a^{nm} \left[a_{nikm} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j a_{njkm} \right] + \frac{g}{Bq}(b + gq)y^m y^n a_{nikm} \\
&- \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) (N - 1)\tilde{k}_k + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) r_k{}^m \tilde{k}_m \\
&\quad + \frac{g}{q}bb^m \left[-\tilde{k}_m r_{ik} - \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j \tilde{k}_m r_{jk} \right] \\
&- \frac{g}{q}y^m \left[\tilde{k}_k r_{im} - \tilde{k}_m r_{ik} + \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j \left(\tilde{k}_k r_{jm} - \tilde{k}_m r_{jk} \right) \right] \\
&- \frac{g}{q}y^n \left[-\tilde{k}_n r_{ki} + \tilde{k}_i r_{nk} + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2)v_i \right) \tilde{k}_m r_{nk} b^m \right] \\
&\quad - \frac{g}{Bq}(b + gq) \frac{g}{q}v_i \left(q^2 \tilde{k}_k - (y\tilde{k})v_k \right), \tag{B.52}
\end{aligned}$$

which can be simplified to read

$$R^m{}_{ikm} + R_i{}^h{}_{hk} = \frac{1}{2}k^2 g^2 (N - 2)r_{ik} + \frac{g^2}{2}b^m \tilde{k}_m \left(r_{ik} - \frac{1}{q^2}v_i v_k \right)$$

$$\begin{aligned}
& + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left[\frac{1}{4} k^2 g^2 (N-2) v_k + \frac{gq}{2} \left(\tilde{k}_m r_k^m - \frac{1}{q^2} \tilde{k}_m v^m v_k - N \tilde{k}_k \right) \right] \\
& + \frac{g}{Bq} (b + gq) \left[\frac{1}{4} k^2 g^2 q^2 \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) - \frac{1}{2} gq (y \tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \right] \\
& + 2a_i^h{}_{hk} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) y^j a_{njkm} a^{nm} + \frac{g}{Bq} (b + gq) y^m y^n a_{nikm} \\
& - \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2) v_i \right) (N-1) \tilde{k}_k + \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2) v_i \right) r_k^m \tilde{k}_m \\
& + \frac{g}{q} b b^m \left[-\tilde{k}_m r_{ik} - \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \tilde{k}_m v_k \right] \\
& - \frac{g}{q} \left[\tilde{k}_k v_i - (y \tilde{k}) r_{ik} + \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left(q^2 \tilde{k}_k - (y \tilde{k}) v_k \right) \right] \\
& - \frac{g}{q} \left[-(y \tilde{k}) r_{ki} + \tilde{k}_i v_k + \frac{g}{q} v_i \tilde{k}_m b^m v_k + \frac{g^2}{B} (q^2 b_i - b v_i) \tilde{k}_m b^m v_k \right]. \tag{B.53}
\end{aligned}$$

Thus, the following proposition is valid.

Proposition B2. *In any Landsberg case of the \mathcal{FF}_g^{PD} -space, the covariantly conserved tensor*

$$\rho_{ij} = \frac{1}{2} (R_i^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2} g_{ij} R^{mn}{}_{nm}$$

is given by the explicit representations (B.53) and (B.39).

Contracting yields

$$\begin{aligned}
y^i y^k \left(R^m{}_{ikm} + R_i^h{}_{hk} \right) &= \frac{1}{2} k^2 g^2 (N-2) q^2 + 2a_i^h{}_{hk} y^i y^k \\
&- gq(N-1)(y \tilde{k}) + gq(v^m \tilde{k}_m) - gbq(b \tilde{k}) - \frac{g}{q} gq^3 \tilde{k}_m b^m, \tag{B.54}
\end{aligned}$$

as well as

$$\begin{aligned}
y^k \left(R^m{}_{ikm} + R_i^h{}_{hk} \right) &= \frac{1}{2} k^2 g^2 (N-2) v_i \\
&+ \frac{g}{B} \left(qb_i - b \frac{v_i}{q} \right) \left[\frac{1}{4} k^2 g^2 (N-2) q^2 - \frac{gq}{2} N (y \tilde{k}) \right]
\end{aligned}$$

$$\begin{aligned}
& +2a_i{}^h{}_{hk}y^k + \frac{g}{B}\left(qb_i - b\frac{v_i}{q}\right)y^jy^ka_{njkm}a^{nm} \\
& -(N-1)\frac{g}{Bq}\left(gq^3b_i + (q^2 + b^2)v_i\right)(y\tilde{k}) + \frac{g}{Bq}\left(gq^3b_i + (q^2 + b^2)v_i\right)(v\tilde{k}) \\
& + \frac{g}{q}b\left[-v_i - \frac{g}{B}\left(qb_i - b\frac{v_i}{q}\right)q^2\right](b\tilde{k}) \\
& - \frac{g}{q}\left[-(y\tilde{k})v_i + \tilde{k}_iq^2 + gqv_i(b\tilde{k}) + \frac{g^2q^2}{B}\left(q^2b_i - bv_i\right)(b\tilde{k})\right]. \tag{B.55}
\end{aligned}$$

Appendix C: Special-case evaluation

In the special case characterized by the warped representation (1.24) of the interval square $(ds)^2$ the vector \tilde{k}_m is a factor of the fundamental vector b_m :

$$\tilde{k}_m = fb_m, \quad f = (b\tilde{k}) \tag{C.1}$$

(see (1.19)), which entails

$$\tilde{k}_mr^m{}_n = 0, \tag{C.2}$$

$$\tilde{k}_mv^m = 0, \tag{C.3}$$

and

$$n_1 = 0 \tag{C.4}$$

(the function n_1 was defined in (B.15)).

Also, it follows that

$$\phi = \phi(z^N), \tag{C.5}$$

so that

$$k = k(z^N). \tag{C.6}$$

Let us put

$$\dot{k} = \frac{dk}{dz^N}, \quad \dot{\phi} = \frac{d\phi}{dz^N} \tag{C.7}$$

and obtain

$$(b\tilde{k}) = \dot{k} + k^2. \tag{C.8}$$

When proceeding in this way, we have

$$r_{ab}(z^A) = \left(\phi(z^N)\right)^2 p_{ab}(z^c) \tag{C.9}$$

and

$$(ds)^2 = (dz^N)^2 + \left(\phi(z^N)\right)^2 p_{ab}(z^c) dz^a dz^b, \tag{C.10}$$

together with

$$\dot{\phi} = k\phi \tag{C.11}$$

and

$$\frac{1}{\phi}\ddot{\phi} = \dot{k} + k^2.$$

The Christoffel symbols (A.111) reduce to

$$s^a{}_{bN} = k\delta_b^a, \quad s^N{}_{bc} = -kr_{bc}, \quad s^a{}_{bc} = \frac{1}{2}p^{ad}\left(\frac{\partial p_{dc}}{\partial z^b} + \frac{\partial p_{bd}}{\partial z^c} - \frac{\partial p_{bc}}{\partial z^d}\right). \quad (\text{C.12})$$

If we calculate from them the Riemannian curvature tensor (see (A.44) in Appendix A), we obtain the components

$$a_a{}^N{}_{bN} = \frac{\partial s^N{}_{aN}}{\partial z^b} - \frac{\partial s^N{}_{ab}}{\partial z^N} + s^h{}_{aN}s^N{}_{hb} - s^h{}_{ab}s^N{}_{hN} = \dot{k}r_{ab} - k^2r_{ab} \quad (\text{C.13})$$

and

$$a_a{}^N{}_{bc} = \frac{\partial s^N{}_{ac}}{\partial z^b} - \frac{\partial s^N{}_{ab}}{\partial z^c} + s^h{}_{ac}s^N{}_{hb} - s^h{}_{ab}s^N{}_{hc} = \frac{\partial s^N{}_{ac}}{\partial z^b} - \frac{\partial s^N{}_{ab}}{\partial z^c} + s^f{}_{ac}s^N{}_{fb} - s^f{}_{ab}s^N{}_{fc}, \quad (\text{C.14})$$

together with

$$a_a{}^e{}_{bc} = \frac{\partial s^e{}_{ac}}{\partial z^b} - \frac{\partial s^e{}_{ab}}{\partial z^c} + s^h{}_{ac}s^e{}_{hb} - s^h{}_{ab}s^e{}_{hc} = P_a{}^e{}_{bc} + s^N{}_{ac}s^e{}_{Nb} - s^N{}_{ab}s^e{}_{Nc}. \quad (\text{C.15})$$

Under these conditions, we obtain

$$a_a{}^N{}_{bN} = (\dot{k} - k^2)r_{ab}, \quad (\text{C.16})$$

$$a_a{}^N{}_{bc} = 0, \quad (\text{C.17})$$

$$a_a{}^e{}_{bc} = P_a{}^e{}_{bc} - k^2(r_{ac}\delta_b^e - r_{ab}\delta_c^e), \quad (\text{C.18})$$

where

$$P_a{}^e{}_{bc} = \frac{\partial p^e{}_{ac}}{\partial z^b} - \frac{\partial p^e{}_{ab}}{\partial z^c} + p^f{}_{ac}p^e{}_{fb} - p^f{}_{ab}p^e{}_{fc}. \quad (\text{C.19})$$

We have

$$a_N{}^h{}_{Nh} = (N-1)(\dot{k} - k^2), \quad (\text{C.20})$$

$$a_a{}^h{}_{Nh} = 0, \quad (\text{C.21})$$

and

$$a_a{}^h{}_{bh} = P_{ab} + (N-2)k^2r_{ab} + (\dot{k} - k^2)r_{ab}. \quad (\text{C.22})$$

In terms of the quantities

$$\xi_1 = -k^2 - \frac{1}{q^2(N-2)}y^a y^b P_{ab}, \quad \xi_2 = -k^2 - \frac{1}{(N-1)(N-2)}a^{ab}P_{ab}, \quad (\text{C.23})$$

we obtain the concise representation

$$y^n y^m a_n^h{}_{mh} = (N-1)(\dot{k} - k^2)b^2 - (N-2)\xi_1 q^2 + (\dot{k} - k^2)q^2. \quad (C.24)$$

Notice that

$$y^N = b. \quad (C.25)$$

Henceforth, we shall use the notation

$$\beta = \dot{k} - k^2 \equiv (b\tilde{k}) - 2k^2. \quad (C.26)$$

The representation

$$\begin{aligned} a_n^i{}_{km} &= r_n^h r_l^i r_k^j r_m^u P_h^l{}_{ju} - k^2 (r_{nm} r_k^i - r_{nk} r_m^i) \\ &+ (b\tilde{k}) (b^i b_m r_{nk} - b^i b_k r_{nm} - b_n b_m r_k^i + b_n b_k r_m^i) \end{aligned} \quad (C.27)$$

is valid, entailing

$$a_n^h{}_{mh} = r_n^i r_m^j P_{ij} - (N-2)k^2 r_{nm} + (b\tilde{k}) r_{nm} + (N-1)(b\tilde{k}) b_n b_m.$$

The scalar μ given by (B.41) reduces now to

$$\mu = -2(N-1)\beta + (N-1)(N-2)\xi_2$$

$$-(N-1)\frac{g^2 q^2}{B}\beta + \frac{gq(b+gq)}{B}(N-2)(\xi_1 + \beta).$$

The contraction (B.54) is got simplified to read

$$\begin{aligned} y^i y^k (R^m{}_{ikm} + R_i^h{}_{hk}) &= \frac{1}{2}k^2 g^2 (N-2)q^2 - 2(N-1)b^2\beta \\ &+ 2(N-2)\xi_1 q^2 - 2q^2\beta - Ngbq(b\tilde{k}) - g^2 q^2 (b\tilde{k}), \end{aligned} \quad (C.28)$$

and (B.55) becomes

$$\begin{aligned} y^k (R^m{}_{ikm} + R_i^h{}_{hk}) &= \frac{1}{2}k^2 g^2 (N-2)v_i \\ &+ \frac{g^2}{B} \left(qb_i - b\frac{v_i}{q} \right) \left[\frac{1}{4}k^2 g(N-2)q^2 - \frac{bq}{2}N(b\tilde{k}) \right] + 2a_i^h{}_{hk} y^k \\ &+ \frac{g}{B} \left(qb_i - b\frac{v_i}{q} \right) y^j y^k a_{njk} a^{nm} \\ &- (N-1)\frac{gb}{Bq} (gq^3 b_i + (q^2 + b^2)v_i) (b\tilde{k}) \\ &- gq\tilde{k}_i - g^2 v_i (b\tilde{k}) - \frac{g^2(b+gq)}{B} (q^2 b_i - bv_i) (b\tilde{k}). \end{aligned} \quad (C.29)$$

The scalar (B.39) reduces to

$$\begin{aligned}
K^2 R^{ni}_{in} = & \frac{1}{4}(N-2)(N-1)k^2 g^2 B + \frac{1}{4}(N-2)\frac{1}{b}k^2 g^3 q(b^2 + gbq + q^2) \\
& - \frac{gq}{b} \left[\frac{1}{4}(N-2)k^2 g^2 q^2 - \frac{1}{2}gbqN(b\tilde{k}) \right] \\
& - 2(N-1)\beta B + (N-1)(N-2)\xi_2 B \\
& - (N-1)g^2 q^2 \beta + (N-2)gq(b + gq)(\xi_1 + \beta), \tag{C.30}
\end{aligned}$$

where ξ_1, ξ_2 are the quantities which were defined in (C.23).

Simplifying yields

$$\begin{aligned}
K^2 R^{ni}_{in} = & -2(N-1)B\beta + (N-1)(N-2) \left(\xi_2 + \frac{1}{4}g^2 k^2 \right) B + \frac{1}{2}(N-2)g^2 q^2 (b\tilde{k}) \\
& + (N-2)gq(b + gq) \left(\xi_1 + \frac{1}{4}g^2 k^2 \right) + (N-2)gbq\beta + 2g^2 q^2 k^2. \tag{C.31}
\end{aligned}$$

Henceforth, we restrict the treatment to the *constant-curvature case*:

$$P_a{}^e{}_{bc} = -\varkappa_1(p_{ac}\delta_b^e - p_{ab}\delta_c^e), \quad \varkappa_1 = -1, 0, 1, \tag{C.32}$$

which can be written as

$$P_a{}^e{}_{bc} = -\varkappa(r_{ac}\delta_b^e - r_{ab}\delta_c^e) \tag{C.33}$$

with

$$\varkappa = \frac{1}{\phi^2} \varkappa_1 \tag{C.34}$$

(see (C.9)). The contracted tensor

$$P_{ab} = P_a{}^e{}_{be} \tag{C.35}$$

equals

$$P_{ab} = (N-2)\varkappa r_{ab}, \tag{C.36}$$

entailing

$$y^a y^b P_{ab} = (N-2)\varkappa q^2, \quad a^{ab} P_{ab} = (N-1)(N-2)\varkappa. \tag{C.37}$$

The quantities (C.23) become

$$\xi_1 = \xi_2 = -k^2 - \varkappa \equiv \xi. \tag{C.38}$$

We obtain

$$a_a{}^h{}_{bh} = -(N-2)\xi r_{ab} + \beta r_{ab} \tag{C.39}$$

and

$$y^n y^m a_n{}^h{}_{mh} = (N-1)\beta b^2 - (N-2)\xi q^2 + \beta q^2, \tag{C.40}$$

together with

$$a_i^h{}_{hk} = -(N-1)\beta b_i b_k + (N-2)\xi r_{ik} - \beta r_{ik}, \quad (\text{C.41})$$

$$-y^j a_j^h{}_{hk} = (N-1)\beta b b_k - (N-2)\xi v_k + \beta v_k, \quad (\text{C.42})$$

and

$$y^m y^n a_{nikm} = -\beta \left[b^2 r_{ik} + q^2 b_i b_k - b b_i v_k - b b_k v_i \right] + \xi (q^2 r_{ik} - v_i v_k). \quad (\text{C.43})$$

With (C.36)–(C.42), the tensor (C.29) reduces to

$$\begin{aligned} y^k \left(R^m{}_{ikm} + R_i^h{}_{hk} \right) &= \frac{1}{2} k^2 g^2 (N-2) v_i \\ -\frac{g^2 q}{B} \left[\frac{1}{4} k^2 g (N-2) q^2 - \frac{b q}{2} N(b\tilde{k}) \right] e_i &- 2(N-1)\beta b b_i + 2(N-2)\xi v_i - 2\beta v_i \\ -(N-2) \frac{g q^3}{B} \xi e_i + \frac{g q}{B} \left[(N-1) b^2 + q^2 \right] \beta e_i & \\ -(N-1) \frac{g q}{B} \left(B b_i + (q^2 + b^2) e_i \right) (b\tilde{k}) & \\ -g q \tilde{k}_i - g^2 v_i (b\tilde{k}) + \frac{g^2 q^2 (b + g q)}{B} (b\tilde{k}) e_i & \\ = \frac{g^2 q b q}{B} \frac{1}{2} N(b\tilde{k}) e_i - 2(N-1)\beta b b_i + 2(N-2)\xi v_i - 2\beta v_i & \\ -(N-2) \frac{g q^3}{B} \xi e_i + \frac{g q}{B} \left[-(N-2) q^2 + g q (b + g q) \right] (b\tilde{k}) e_i & \\ -N g q b_i (b\tilde{k}) - g^2 v_i (b\tilde{k}) - 2(N-1) \frac{g q b^2}{B} k^2 e_i - 2 \frac{g q^3}{B} k^2 e_i & \\ = \frac{g^2 q b q}{B} \frac{1}{2} N(b\tilde{k}) e_i - (N-2) \frac{g q^3}{B} \xi e_i + \frac{g q}{B} \left[-(N-2) q^2 + g q (b + g q) \right] (b\tilde{k}) e_i & \\ + 2(N-2) \xi_9 \frac{q^2}{b} e_i - 2\beta \frac{q^2}{b} e_i - g^2 (b\tilde{k}) \frac{q^2}{b} e_i - 2(N-1) \beta b b_i - N g q (b\tilde{k}) b_i & \end{aligned}$$

$$+2(N-2)\xi_9\frac{q^2}{b}b_i - 2\beta\frac{q^2}{b}b_i - g^2(b\tilde{k})\frac{q^2}{b}b_i - 2(N-1)\frac{gqb^2}{B}k^2e_i - 2\frac{gq^3}{B}k^2e_i,$$

where

$$\xi_9 = \xi + \frac{1}{4}g^2k^2,$$

or, by virtue of (C.38),

$$\xi_9 = -\left(1 - \frac{g^2}{4}\right)k^2 - \varkappa. \quad (\text{C.44})$$

We have used the vector

$$e_i = -b_i + \frac{b}{q^2}v_i. \quad (\text{C.45})$$

If we remind

$$y_i = \left(v_i + (b + gq)b_i\right)\frac{K^2}{B}$$

(see (A.23)), we may write the equality

$$y_i = (q^2e_i + Bb_i)\frac{K^2}{bB}.$$

This yields

$$\begin{aligned} y^k \left(R^m{}_{ikm} + R_i{}^h{}_{hk} \right) &= \frac{g^2q}{B} \frac{bq}{2} N(b\tilde{k})e_i \\ &- (N-2)\frac{gq^3}{B}\xi_9e_i - (N-2)\frac{gq^3}{B}(b\tilde{k})e_i - \frac{g^2q^4}{Bb}(b\tilde{k})e_i \\ &+ 2(N-2)\xi_9\frac{q^2}{b}e_i - 2(b\tilde{k})\frac{q^2(b^2 + gbq + q^2)}{Bb}e_i \\ &+ 2(N-1)(b\tilde{k})b\frac{q^2}{B}e_i + Ngq(b\tilde{k})\frac{q^2}{B}e_i \\ &- 2(N-2)\xi_9\frac{q^2}{b}\frac{q^2}{B}e_i + 2\beta\frac{q^2}{b}\frac{q^2}{B}e_i + g^2(b\tilde{k})\frac{q^2}{b}\frac{q^2}{B}e_i + R_9\frac{1}{K^2}y_i \\ &- 2(N-1)\frac{gqb^2}{B}k^2e_i - 2\frac{gq^3}{B}k^2e_i, \end{aligned}$$

where

$$R_9 = -2(N-1)\beta b^2 - Ngq(b\tilde{k})b + 2(N-2)\xi_9q^2 - 2\beta q^2 - g^2(b\tilde{k})q^2,$$

or

$$R_9 = -2(N-2)\beta b^2 - (N-2)gq(b\tilde{k})b + 2(N-2)\xi_9q^2 - 2(b\tilde{k})B - g^2(b\tilde{k})q^2 + 4(b^2 + q^2)k^2.$$

In this way we arrive at the equality

$$y^k \left(R^m_{ikm} + R_i^h{}_{hk} \right) = (N-2)I_4 q e_i + N \frac{g^2 b q^2}{2B} (b\tilde{k}) e_i + 4 \frac{g q^3}{B} k^2 e_i$$

$$- 2(N-1) \frac{g q b^2}{B} k^2 e_i - 2 \frac{g q^3}{B} k^2 e_i + R_9 \frac{1}{K^2} y_i \quad (\text{C.46})$$

with

$$I_4 = -\frac{g q^2}{B} \xi_9 + 2 \xi_9 \frac{q}{b} + 2(b\tilde{k}) \frac{b q}{B} - 2 \xi_9 \frac{q^3}{B b},$$

or

$$I_4 = \frac{g q^2}{B} \xi_9 + 2 \left(\xi_9 + (b\tilde{k}) \right) \frac{b q}{B}. \quad (\text{C.47})$$

The scalar (C.31) takes on the form

$$K^2 R^{ni}{}_{in} = -2(N-1)\beta B + (N-1)(N-2)\xi_9 B$$

$$+ \frac{1}{2}(N-2)g^2 q^2 (b\tilde{k}) + (N-2)g q (b + g q) \xi_9 + (N-2)g b q \beta + 2g^2 q^2 k^2. \quad (\text{C.48})$$

Now we calculate the quantity

$$\epsilon = \frac{1}{2}(R_9 - K^2 R^{ni}{}_{in}). \quad (\text{C.49})$$

With (C.48), we find

$$R_9 - K^2 R^{ni}{}_{in} = (N-2)E_4 - g^2 q^2 (b\tilde{k}) - 2k^2 B + 4q^2 k^2 - 2g^2 q^2 k^2, \quad (\text{C.50})$$

where

$$E_4 = 2 \left(1 - \frac{g^2}{4} \right) (b\tilde{k}) q^2 + 2 \xi_9 q^2 - (N-1) \xi_9 B - g q (b + g q) \xi_9 - 4q^2 k^2 - 2g b q k^2. \quad (\text{C.51})$$

Thus we get

$$\epsilon = \frac{1}{2}(N-2)E_4 - \frac{1}{2}g^2 q^2 (b\tilde{k}) - 2k^2 B + 2(b^2 + q^2)k^2 - g^2 q^2 k^2,$$

or

$$\epsilon = \frac{1}{2}(N-2)E_4 - \frac{1}{2}g^2 q^2 (b\tilde{k}) - 2g b q k^2 - g^2 q^2 k^2. \quad (\text{C.52})$$

Now we are able to evaluate the explicit expression of the vector

$$\rho_i := \rho_{ij} y^j.$$

From (A.65) we get

$$2\rho_i = y^k \left(R^m_{ikm} + R_i^h{}_{hk} \right) - y_i R^{mn}{}_{nm}. \quad (\text{C.53})$$

We obtain

$$2\rho_i = (N-2)I_4 q e_i + \frac{g^2 b q^2}{2B} N (b\tilde{k}) e_i + 4 \frac{g q^3}{B} k^2 e_i$$

$$-2(N-1)\frac{gqb^2}{B}k^2e_i - 2\frac{gq^3}{B}k^2e_i + \epsilon\frac{1}{K^2}y_i. \quad (\text{C.54})$$

If we apply here the formula (A.36) which introduces the vector A_i , we can obtain the expansion

$$\rho_i = \epsilon\frac{1}{K^2}y_i - r_1qA_i, \quad (\text{C.55})$$

where

$$r_1 = \frac{b}{2gK} \left[\frac{2(N-2)}{N}P_4 + g^2(b\tilde{k}) + \frac{4gq}{Nb}k^2 - \frac{4(N-1)}{N}g\frac{b}{q}k^2 \right] \quad (\text{C.56})$$

with

$$P_4 = g\frac{q}{b}\xi_9 + 2\left(\xi_9 + (b\tilde{k})\right) \equiv \frac{B}{q}I_4. \quad (\text{C.57})$$

Raising the index yields the expansion

$$\rho^i = \epsilon\frac{1}{K^2}y^i - r_1qA^i. \quad (\text{C.58})$$

We may apply here the representation (A.37) of the vector A^i , obtaining

$$\rho^i = \epsilon\frac{1}{K^2}y^i - s_1\frac{1}{K^2}\left[Bb^i - (b + gq)y^i\right], \quad (\text{C.59})$$

where

$$s_1 = \frac{Nb}{4} \left[\frac{2(N-2)}{N}P_4 + g^2(b\tilde{k}) + \frac{8gq}{Nb}k^2 \right]. \quad (\text{C.60})$$

In another convenient form,

$$\rho^i = -s_1\frac{B}{K^2}b^i + \frac{P}{K^2}y^i, \quad (\text{C.61})$$

where

$$P = \epsilon + s_1(b + gq). \quad (\text{C.62})$$

Let us apply the *osculation procedure*, specifying the variable y^h according to

$$y^h = b^h(x). \quad (\text{C.63})$$

Noting that

$$q|_{y^h=b^h} = 0, \quad b|_{y^h=b^h} = 1,$$

from (C.52) we get

$$\epsilon|_{y^h=b^h} = -\frac{1}{2}(N-1)(N-2)\xi_9 \quad (\text{C.64})$$

and (C.60) yields us

$$s_1|_{y^h=b^h} = (N-2)\xi_9 + \left(N-2 + \frac{N}{4}g^2\right)(b\tilde{k}). \quad (\text{C.65})$$

We have

$$\rho^i|_{y^h=b^h} = \epsilon|_{y^h=b^h}b^i \quad (\text{C.66})$$

together with

$$P|_{y^h=b^h} = \mathcal{P}, \quad (\text{C.67})$$

where

$$\mathcal{P} = \epsilon|_{y^h=b^h} + s1|_{y^h=b^h}$$

(see (C.62)), that is,

$$\mathcal{P} = -\frac{1}{2}(N-2)(N-3)\xi_9 + \left(N-2 + \frac{N}{4}g^2\right)(b\tilde{k}). \quad (\text{C.68})$$

From (C.61) we infer

$$\left(\frac{\partial(\frac{\rho^i}{b})}{\partial y^i}\right)|_{y^h=b^h} = \left(\frac{\partial(\frac{Py^i}{bK^2})}{\partial y^i}\right)|_{y^h=b^h} = -\left(\frac{P}{bK^2}\right)|_{y^h=b^h} + N\left(\frac{P}{bK^2}\right)|_{y^h=b^h} = (N-1)\mathcal{P},$$

that is,

$$\left(\frac{\partial(\rho^i/b)}{\partial y^i}\right)|_{y^h=b^h} = (N-1)\mathcal{P}. \quad (\text{C.69})$$

All the conditions which have underlined the equality (A.108) of Appendix A are fulfilled, so that we are entitled to write

$$\nabla_i \rho_{\{b\}}^i = (N-1)k\mathcal{P}, \quad (\text{C.70})$$

or, if we apply (A.109),

$$b^i \partial_i \gamma_{\{b\}} + (N-1)k\gamma_{\{b\}} = (N-1)k\mathcal{P}. \quad (\text{C.71})$$

From (C.66) and (A.105) it follows that

$$\gamma_{\{b\}} = \epsilon|_{y^h=b^h}. \quad (\text{C.72})$$

Appendix D: Evaluation of the covariantly conserved tensor ρ^i_k

Let us evaluate the sum tensor $R^m_{ikm} + R_i^h{}_{hk}$, assuming the special-case relations (C.1)–(C.12) to be valid.

From (B.53) it follows that

$$\begin{aligned} R^m_{ikm} + R_i^h{}_{hk} &= \frac{1}{2}k^2g^2(N-2)r_{ik} + \frac{g^2}{2}\left(r_{ik} - \frac{1}{q^2}v_iv_k\right)(b\tilde{k}) \\ &\quad - \frac{gq}{B}\left[\frac{1}{4}k^2g^2(N-2)v_k - \frac{gq}{2}N\tilde{k}_k\right]e_i \\ &\quad + \frac{g}{Bq}(b+gq)\left[\frac{1}{4}k^2g^2q^2\left(r_{ik} - \frac{1}{q^2}v_iv_k\right) - \frac{1}{2}gbq(b\tilde{k})\left(r_{ik} - \frac{1}{q^2}v_iv_k\right)\right] \end{aligned}$$

$$\begin{aligned}
& +2a_i{}^h{}_{hk} - \frac{gq}{B}y^ja_j{}^h{}_{hk}e_i + \frac{g}{Bq}(b+gq)y^my^na_{nikm} \\
& -(N-1)\frac{g}{Bq}\left(gq^3b_i+(q^2+b^2)v_i\right)\tilde{k}_k \\
& +\frac{g}{q}b\left[-r_{ik}+\frac{gq}{B}e_iv_k\right](b\tilde{k})+\frac{2gb}{q}(b\tilde{k})r_{ik}-\frac{g}{q}\left[\tilde{k}_kv_i-\frac{gq}{B}e_i\left(q^2\tilde{k}_k-(y\tilde{k})v_k\right)\right] \\
& -\frac{g}{q}\left[\tilde{k}_iv_k+\frac{g}{q}v_i(b\tilde{k})v_k-\frac{g^2q^2}{B}e_i(b\tilde{k})v_k\right].
\end{aligned}$$

Cancel similar terms and make insertion of the Riemannian curvature tensor contractions (C.41)–(C.43) related to the isotropic case. We obtain

$$\begin{aligned}
R^m{}_{ikm}+R_i{}^h{}_{hk}&=\frac{1}{2}k^2g^2(N-2)r_{ik}+\frac{g^2}{2}\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)(b\tilde{k}) \\
& -\frac{gq}{B}\left[\frac{1}{4}k^2g^2(N-2)v_k-\frac{gq}{2}N\tilde{k}_k\right]e_i \\
& +\frac{g}{Bq}(b+gq)\left[\frac{1}{4}k^2g^2q^2\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)-\frac{1}{2}gbq(b\tilde{k})\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)\right] \\
& -2(N-1)\beta b_ib_k+2(N-2)\xi r_{ik}-2\beta r_{ik} \\
& +\frac{gq}{B}\left((N-1)\beta b_bk-(N-2)\xi v_k+\beta v_k\right)e_i \\
& +\frac{g}{Bq}(b+gq)\left[-\beta\left[b^2r_{ik}+q^2b_ib_k-bb_iv_k-bb_kv_i\right]+\xi(q^2r_{ik}-v_iv_k)\right] \\
& -(N-1)\frac{g}{Bq}\left(gq^3b_i+(q^2+b^2)v_i\right)\tilde{k}_k \\
& +\frac{gb}{q}(b\tilde{k})r_{ik}-\frac{g}{q}\left[v_i-\frac{gq^3}{B}e_i\right]\tilde{k}_k \\
& -\frac{g}{q}\left[\tilde{k}_iv_k+\frac{g}{q}v_i(b\tilde{k})v_k-\frac{g^2q^2}{B}e_i(b\tilde{k})v_k\right],
\end{aligned}$$

where β is the quantity (C.26).

Here, the variable ξ can conveniently be replaced by the variable $\xi_9 = \xi + \frac{1}{4}g^2k^2$ (see (C.44)). Following in this way, we get

$$\begin{aligned}
R^m_{ikm} + R_i{}^h{}_{hk} &= \frac{g^2}{2} \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) (b\tilde{k}) + \frac{gq}{B} \frac{gq}{2} N \tilde{k}_k e_i \\
&\quad - \frac{1}{2} \frac{g^2}{B} (B - q^2) (b\tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\
&\quad - 2(N-1)\beta b_i b_k + 2(N-2)\xi_9 r_{ik} - 2\beta r_{ik} \\
&\quad + \frac{gq}{B} \left((N-1)\beta b b_k - (N-2)\xi_9 v_k + \beta v_k \right) e_i \\
&\quad - \frac{g}{Bq} (b + gq) \left[b^2 r_{ik} + q^2 b_i b_k - b b_i v_k - b b_k v_i \right] \beta \\
&\quad + \frac{gq}{B} (b + gq) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \xi_9 - (N-1) \frac{g}{Bq} \left(gq^3 b_i + (q^2 + b^2) v_i \right) \tilde{k}_k \\
&\quad + \frac{gb}{q} (b\tilde{k}) r_{ik} - \frac{g}{q} \left[v_i - \frac{gq^3}{B} e_i \right] (b\tilde{k}) b_k - \frac{g}{q} \left[b_i + \frac{g}{q} v_i - \frac{g^2 q^2}{B} e_i \right] (b\tilde{k}) v_k. \tag{D.1}
\end{aligned}$$

Applying

$$e_i = -b_i + \frac{b}{q^2} v_i$$

leads to

$$\begin{aligned}
R^m_{ikm} + R_i{}^h{}_{hk} &= \frac{gq}{B} \frac{gq}{2} N \tilde{k}_k e_i + \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\
&\quad - 2(N-1)\beta b_i b_k + 2(N-2)\xi_9 r_{ik} - 2\beta r_{ik} \\
&\quad + \frac{gq}{B} \left((N-1)\beta b b_k - (N-2)\xi_9 v_k + \beta v_k \right) e_i \\
&\quad - \frac{g}{qb} \left[b^2 r_{ik} + q^2 b_i b_k - b b_i v_k - b b_k v_i \right] \beta \\
&\quad + \frac{gq}{Bb} \left[b^2 r_{ik} + q^2 b_i b_k - b b_i v_k - b b_k v_i \right] \beta
\end{aligned}$$

$$\begin{aligned}
& +\frac{gq}{B}(b+gq)\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)\xi_9 \\
& -(N-1)\frac{g}{Bq}\frac{q^2}{b}B(\tilde{b}k)b_ib_k-(N-1)\frac{gq}{Bb}(q^2+b^2)(\tilde{b}k)b_ke_i \\
& +\frac{gb}{q}(\tilde{b}k)r_{ik}-\frac{g}{q}\left[\frac{q^2}{b}b_i+\frac{q^2}{b}e_i-\frac{gq^3}{B}e_i\right](\tilde{b}k)b_k \\
& -\frac{g}{q}(\tilde{b}k)b_iv_k-\frac{g^2}{q^2}(\tilde{b}k)v_iv_k+\frac{g^3q}{B}(\tilde{b}k)e_iv_k.
\end{aligned}$$

Let us perform appropriate cancellation:

$$\begin{aligned}
R^m{}_{ikm}+R_i{}^h{}_{hk}&=\frac{gq}{B}\frac{gq}{2}N(\tilde{b}k)b_ke_i \\
& +\frac{1}{2}\frac{g^2q^2}{B}(\tilde{b}k)\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)+\frac{gq}{B}(b+gq)\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)\xi_9 \\
& -2(N-1)\beta b_ib_k+2(N-2)\xi_9r_{ik}-2\beta r_{ik} \\
& +\frac{gq}{B}\left((N-1)\beta bb_k-(N-2)\xi_9v_k+\beta v_k\right)e_i \\
& -\frac{g}{qb}\left[q^2b_ib_k-bb_kv_i\right]\beta \\
& +\frac{gq}{Bb}\left[b^2r_{ik}+q^2b_ib_k-bb_kv_k-bb_kv_i\right]\beta \\
& -N\frac{gq}{b}(\tilde{b}k)b_ib_k-N\frac{gq}{Bb}(q^2+b^2)(\tilde{b}k)b_ke_i \\
& -\frac{g^2}{q^2}(\tilde{b}k)v_iv_k+\frac{g^3q}{B}(\tilde{b}k)e_iv_k \\
& =\frac{N}{2}\frac{g^2q^2}{B}(\tilde{b}k)b_ke_i-N\frac{gq}{Bb}(q^2+b^2)(\tilde{b}k)b_ke_i \\
& +\frac{1}{2}\frac{g^2q^2}{B}(\tilde{b}k)\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)+\frac{gq}{B}(b+gq)\left(r_{ik}-\frac{1}{q^2}v_iv_k\right)\xi_9
\end{aligned}$$

$$\begin{aligned}
& -2(N-1)(\tilde{b}\tilde{k})b_ib_k + 2(N-2)\xi_9 r_{ik} - 2(\tilde{b}\tilde{k})r_{ik} + \frac{gbq}{B}(\tilde{b}\tilde{k})r_{ik} \\
& + \frac{gq}{B} \left((N-1)(\tilde{b}\tilde{k})bb_k - (N-2)\xi_9 v_k + (\tilde{b}\tilde{k})\frac{q^2}{b}b_k + (\tilde{b}\tilde{k})\frac{q^2}{b}e_k \right) e_i \\
& + (\tilde{b}\tilde{k})\frac{gq}{b}b_k e_i - \frac{gq^3}{Bb} \left[b_ib_k + b_ie_k + b_ke_i \right] (\tilde{b}\tilde{k}) \\
& - N\frac{gq}{b}(\tilde{b}\tilde{k})b_k \left(b_i + \frac{q^2}{B}e_i - \frac{q^2}{B}e_i \right) \\
& - \frac{g^2}{b}(\tilde{b}\tilde{k})v_kb_i - \frac{g^2}{b}(\tilde{b}\tilde{k})v_ke_i + \frac{g^3q}{B}(\tilde{b}\tilde{k})e_iv_k, \tag{D.2}
\end{aligned}$$

or

$$\begin{aligned}
& R^m{}_{ikm} + R_i{}^h{}_{hk} = \frac{N}{2} \frac{g^2 q^2}{B} (\tilde{b}\tilde{k}) b_k e_i \\
& + \frac{1}{2} \frac{g^2 q^2}{B} (\tilde{b}\tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) + \frac{gq}{B} (b + gq) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \xi_9 \\
& + \left[2(N-2)\xi_9 - 2(\tilde{b}\tilde{k}) + \frac{gbq}{B}(\tilde{b}\tilde{k}) \right] \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\
& + \frac{1}{b} \left[2(N-2)\xi_9 - 2(\tilde{b}\tilde{k}) \right] v_k (b_i + e_i) + \frac{gq}{B} (\tilde{b}\tilde{k}) v_k (e_i + b_i) \\
& - 2(N-1)(\tilde{b}\tilde{k})b_k \left(b_i + \frac{q^2}{B}e_i - \frac{q^2}{B}e_i \right) - (N-2)\frac{gq}{B}\xi_9 v_k e_i \\
& - \frac{gbq}{B}(\tilde{b}\tilde{k})b_k e_i + \frac{gq^3}{Bb}(\tilde{b}\tilde{k})e_k e_i + (\tilde{b}\tilde{k})\frac{gq}{b}b_k e_i \\
& - \frac{gq^3}{Bb}(\tilde{b}\tilde{k}) \left[b_k + e_k \right] b_i - N\frac{gq}{b}(\tilde{b}\tilde{k})b_k \left(b_i + \frac{q^2}{B}e_i \right) \\
& - \frac{g^2}{b}(\tilde{b}\tilde{k})v_k \left(b_i + \frac{q^2}{B}e_i - \frac{q^2}{B}e_i \right) - \frac{g^2}{Bb}(b^2 + q^2)(\tilde{b}\tilde{k})e_iv_k.
\end{aligned}$$

The last expression can conveniently written as

$$R^m{}_{ikm} + R_i{}^h{}_{hk} = \frac{N}{2} \frac{g^2 q^2}{B} (\tilde{b}\tilde{k}) b_k e_i$$

$$\begin{aligned}
& + \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) + \frac{gq}{B} (b + gq) \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \xi_9 \\
& + \left[2(N-2)\xi_9 - 2(b\tilde{k}) + \frac{gbq}{B} (b\tilde{k}) \right] \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) \\
& + \frac{1}{b} \left[2(N-2)\xi_9 - 2(b\tilde{k}) \right] v_k \left[e_i + b_i + \frac{q^2}{B} e_i - \frac{q^2}{B} e_i \right] + \frac{gq}{B} (b\tilde{k}) v_k e_i \\
& - 2(N-1)(b\tilde{k}) b_k \left(b_i + \frac{q^2}{B} e_i - \frac{q^2}{B} e_i \right) - (N-2) \frac{gq}{B} \xi_9 v_k e_i \\
& + \frac{gq^3}{Bb} (b\tilde{k}) e_k e_i + \frac{gq^2}{bB} (q + gb) (b\tilde{k}) b_k e_i \\
& - N \frac{gq}{b} (b\tilde{k}) \left(b_i + \frac{q^2}{B} e_i \right) b_k - \frac{g^2}{b} (b\tilde{k}) \left(b_i + \frac{q^2}{B} e_i \right) v_k - \frac{g^2 b}{B} (b\tilde{k}) e_i v_k. \tag{D.3}
\end{aligned}$$

Next, we use the equality

$$y_i = (q^2 e_i + B b_i) \frac{K^2}{bB} \tag{D.4}$$

(see (A.23) in Appendix A and (C.45) in Appendix C), obtaining

$$R^m{}_{ikm} + R_i{}^h{}_{hk} = 2X \left(r_{ik} - \frac{1}{q^2} v_i v_k \right) + 2Y_k e_i + \frac{2}{K^2} Z_k y_i \tag{D.5}$$

with

$$X = (N-2)\xi_9 - (b\tilde{k}) + \frac{1}{2} \frac{gbq}{B} (b\tilde{k}) + \frac{1}{4} \frac{g^2 q^2}{B} (b\tilde{k}) + \frac{1}{2} \frac{gq}{B} (b + gq) \xi_9 \tag{D.6}$$

and

$$Z_k = \left[(N-2)\xi_9 - (b\tilde{k}) \right] v_k - (N-1)b(b\tilde{k})b_k - \frac{N}{2} gq(b\tilde{k})b_k - \frac{1}{2} g^2 (b\tilde{k}) v_k.$$

Insert here $v_k = (q^2/b)(e_k + b_k)$:

$$\begin{aligned}
Z_k &= \left[(N-2)\xi_9 - (b\tilde{k}) \right] \frac{q^2}{b} e_k + \left[\left[(N-2)\xi_9 - (b\tilde{k}) \right] \frac{q^2}{b} - (N-1)b(b\tilde{k}) \right] b_k \\
&\quad - \frac{N}{2} gq(b\tilde{k})b_k - \frac{1}{2} g^2 (b\tilde{k}) \frac{q^2}{b} e_k - \frac{1}{2} g^2 (b\tilde{k}) \frac{q^2}{b} b_k.
\end{aligned}$$

Apply (D.4):

$$Z_k = \left[(N-2)\xi_9 - (b\tilde{k}) \right] \frac{q^2}{b} e_k + \left[\left[(N-2)\xi_9 - (b\tilde{k}) \right] q^2 - (N-1)b^2(b\tilde{k}) \right] \frac{1}{K^2} y_k$$

$$\begin{aligned}
& - \left[\left[(N-2)\xi_9 - (b\tilde{k}) \right] q^2 - (N-1)b^2(b\tilde{k}) \right] \frac{q^2}{bB} e_k \\
& - \frac{N}{2} gq(b\tilde{k}) \left(\frac{b}{K^2} y_k - \frac{q^2}{B} e_k \right) - \frac{1}{2} g^2(b\tilde{k}) \frac{q^2}{b} e_k - \frac{1}{2} g^2(b\tilde{k}) \frac{q^2}{b} \left(\frac{b}{K^2} y_k - \frac{q^2}{B} e_k \right),
\end{aligned}$$

so that

$$Z_k = Z_{\{y\}} \frac{1}{K^2} y_k + Z_{\{e\}} \frac{q^2}{B} e_k \quad (\text{D.7})$$

with

$$Z_{\{y\}} = \left[(N-2)\xi_9 - (b\tilde{k}) \right] q^2 - (N-1)b^2(b\tilde{k}) - \frac{N}{2} gbq(b\tilde{k}) - \frac{1}{2} g^2(b\tilde{k}) q^2 \quad (\text{D.8})$$

and

$$\begin{aligned}
Z_{\{e\}} &= \left[(N-2)\xi_9 - (b\tilde{k}) \right] \frac{B}{b} - \left[\left[(N-2)\xi_9 - (b\tilde{k}) \right] q^2 - (N-1)b^2(b\tilde{k}) \right] \frac{1}{b} \\
&\quad + \frac{N}{2} gq(b\tilde{k}) - \frac{1}{2} g^2(b\tilde{k}) \frac{B}{b} + \frac{1}{2} g^2(b\tilde{k}) \frac{q^2}{b},
\end{aligned}$$

which can be written as

$$Z_{\{e\}} = (N-2)\xi_9 \frac{B-q^2}{b} + (N-2)b(b\tilde{k}) + \frac{1}{2}(N-2)gq(b\tilde{k}) - \frac{1}{2}g^2(b\tilde{k}) \frac{B}{b} + \frac{1}{2}g^2(b\tilde{k}) \frac{q^2}{b}. \quad (\text{D.9})$$

Also, we get

$$\begin{aligned}
Y_k &= \frac{N}{4} \frac{g^2 q^2}{B} (b\tilde{k}) b_k + (N-2) \frac{b+gq}{B} \xi_9 v_k - \frac{b}{B} (b\tilde{k}) v_k - \frac{gq}{B} (b\tilde{k}) v_k \\
&\quad + \frac{1}{2} \frac{gq}{B} (b\tilde{k}) v_k + (N-1)(b\tilde{k}) \frac{q^2}{B} b_k \\
&\quad - \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 v_k + \frac{gq^3}{2Bb} (b\tilde{k}) e_k + \frac{1}{2} \frac{gq^3}{bB} (b\tilde{k}) b_k - \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) e_k,
\end{aligned}$$

or

$$\begin{aligned}
Y_k &= \frac{N}{4} \frac{g^2 q^2}{B} (b\tilde{k}) b_k + (N-2) \frac{b+gq}{B} \xi_9 v_k - \frac{b}{B} (b\tilde{k}) v_k \\
&\quad + (N-1)(b\tilde{k}) \frac{q^2}{B} b_k - \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 v_k - \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) e_k.
\end{aligned}$$

Inserting here $v_k = (q^2/b)(e_k + b_k)$ yields

$$\begin{aligned}
Y_k &= \frac{N}{4} \frac{g^2 q^2}{B} (b\tilde{k}) b_k + (N-2) \frac{b+gq}{B} \xi_9 \frac{q^2}{b} e_k + (N-2) \frac{b+gq}{B} \xi_9 \frac{q^2}{b} b_k - \frac{q^2}{B} (b\tilde{k}) e_k - \frac{q^2}{B} (b\tilde{k}) b_k \\
&\quad + (N-1)(b\tilde{k}) \frac{q^2}{B} b_k - \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 \frac{q^2}{b} e_k - \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 \frac{q^2}{b} b_k - \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) e_k.
\end{aligned}$$

or

$$Y_k = \frac{N}{4} \frac{g^2 q^2}{B} (b\tilde{k}) b_k + (N-2) \frac{q^2}{B} \xi_9 e_k + (N-2) \frac{q^2}{B} \xi_9 b_k - \frac{q^2}{B} (b\tilde{k}) e_k$$

$$+ (N-2) (b\tilde{k}) \frac{q^2}{B} b_k + \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 \frac{q^2}{b} e_k + \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 \frac{q^2}{b} b_k - \frac{1}{2} \frac{g^2 q^2}{B} (b\tilde{k}) e_k,$$

so that,

$$Y_k = \frac{q^2}{B} Y_{\{y\}} \frac{1}{K^2} y_k + \frac{q^2}{B} Y_{\{e\}} e_k \quad (\text{D.10})$$

with

$$Y_{\{y\}} = (N-2) b \xi_9 + n_7 (b\tilde{k}) b + \frac{1}{2} (N-2) g q \xi_9 \quad (\text{D.11})$$

and

$$Y_{\{e\}} = \left((N-2) \xi_9 - (b\tilde{k}) \right) - (N-2) \frac{q^2}{B} \xi_9$$

$$- n_7 (b\tilde{k}) \frac{q^2}{B} + \frac{1}{2} (N-2) \frac{gq}{b} \xi_9 - \frac{1}{2} (N-2) \frac{gq}{B} \xi_9 \frac{q^2}{b} - \frac{1}{2} g^2 (b\tilde{k}),$$

or

$$Y_{\{e\}} = (N-2) \xi_9 - (b\tilde{k}) - (N-2) \frac{q^2}{B} \xi_9 - n_7 (b\tilde{k}) \frac{q^2}{B} + \frac{1}{2} (N-2) \frac{gq}{B} (b + gq) \xi_9 - \frac{1}{2} g^2 (b\tilde{k}), \quad (\text{D.12})$$

where we have introduced the notation

$$n_7 = N - 2 + \frac{N}{4} g^2. \quad (\text{D.13})$$

Since $y^k e_k = 0$, from (D.7) and (D.10) it ensues that

$$Z_k y^k = Z_{\{y\}}, \quad Y_k y^k = \frac{q^2}{B} Y_{\{y\}}. \quad (\text{D.14})$$

The function (D.11) obeys the equality

$$Y_{\{y\}} = s_1, \quad (\text{D.15})$$

where the right-hand part is the function (C.60) of Appendix C.

We may use the angular metric tensor

$$h_{ij} = \frac{1}{A_l A^l} A_i A_j + \left(r_{ij} - \frac{1}{q^2} v_i v_j \right) \frac{K^2}{B} \quad (\text{D.16})$$

and the contracted Cartan tensor

$$A_i = -\frac{NK}{2} g \frac{q}{B} e_i, \quad A_h A^h = \frac{N^2}{4} g^2, \quad (\text{D.17})$$

obtaining

$$R^m{}_{ikm} + R_i{}^h{}_{hk} = 2X \frac{B}{K^2} \left[h_{ki} - \frac{1}{A_l A^l} A_i A_k \right] - \frac{2B}{NKgq} 2Y_k A_i + \frac{2}{K^2} Z_k y_i.$$

With this result, we obtain the tensor ρ^i_k to read explicitly as

$$\rho^i_k = X \frac{B}{K^2} \left[h^i_k - \frac{1}{A_l A^l} A^i A_k \right] - \frac{2B}{NKgq} Y_k A^i + \frac{1}{K^2} Z_k y^i - \frac{1}{2} \delta^i_k R^{nh}_{hn}. \quad (\text{D.18})$$

Here,

$$h^i_k = \delta^i_k - \frac{1}{K^2} y^i y_k, \quad (\text{D.19})$$

so that

$$\begin{aligned} \rho^i_k = & \frac{1}{2K^2} M_7 \left[h^i_k - \frac{1}{A_l A^l} A^i A_k \right] - \frac{1}{2} \frac{1}{A_l A^l} A^i A_k R^{nh}_{hn} \\ & - \frac{2B}{NKgq} Y_k A^i + \frac{1}{K^2} Z_k y^i - \frac{1}{2} \frac{1}{K^2} y^i y_k R^{nh}_{hn}, \end{aligned} \quad (\text{D.20})$$

where

$$M_7 = 2BX - K^2 R^{nh}_{hn}. \quad (\text{D.21})$$

From (C.48) we know that

$$\begin{aligned} K^2 R^{ni}_{in} = & -2(N-1)(b\tilde{k})B + (N-1)(N-2)B\xi_9 \\ & + \frac{1}{2}(N-2)g^2q^2(b\tilde{k}) + (N-2)gq(b+gq)\xi_9 + (N-2)gbq(b\tilde{k}). \end{aligned} \quad (\text{D.22})$$

By the help of (D.6) and (D.22), the scalar (D.21) reduces to

$$\begin{aligned} M_7 = & 2(N-2)(b\tilde{k})B - (N-3)(N-2)B\xi_9 \\ & - \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) - (N-3)gq(b+gq)\xi_9 - (N-3)gbq(b\tilde{k}). \end{aligned}$$

In terms of the function \mathcal{P} (see (C.68) in Appendix C) this scalar can be written as

$$\begin{aligned} M_7 = & 2B\mathcal{P} - \frac{N}{2}g^2B(b\tilde{k}) \\ & - \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) - (N-3)gq(b+gq)\xi_9 - (N-3)gbq(b\tilde{k}). \end{aligned} \quad (\text{D.23})$$

Using (D.7) and (D.11) in (D.20) leads to the expansion

$$\begin{aligned} \rho^i_k = & \frac{1}{2K^2} M_7 \left[h^i_k - \frac{1}{A_l A^l} A^i A_k \right] - \frac{1}{2} R^{nh}_{hn} \frac{1}{A_l A^l} A^i A_k + \frac{B}{K^2} Y_{\{e\}} \frac{1}{A_l A^l} A^i A_k \\ & - \frac{2q}{NKg} \frac{1}{K^2} \left(Y_{\{y\}} y_k A^i + Z_{\{e\}} A_k y^i \right) + \frac{1}{K^2} \frac{1}{K^2} Z_{\{y\}} y^i y_k - \frac{1}{2} \frac{1}{K^2} R^{nh}_{hn} y^i y_k, \end{aligned}$$

or

$$\rho^i_k = \frac{1}{2K^2} M_7 h_k^i + \frac{1}{2K^2} M_8 \frac{1}{A_l A^l} A^i A_k - \frac{2q}{NKg} \frac{1}{K^2} \left(Y_{\{y\}} y_k A^i + Z_{\{e\}} A_k y^i \right) + \frac{1}{2K^2} M_9 \frac{1}{K^2} y^i y_k, \quad (\text{D.24})$$

where

$$M_8 = -M_7 - K^2 R^{nh}_{hn} + 2BY_{\{e\}} = -2BX + 2BY_{\{e\}} \quad (\text{D.25})$$

and

$$M_9 = 2Z_{\{y\}} - K^2 R^{nh}_{hn}. \quad (\text{D.26})$$

Simple direct calculations yield

$$M_8 = -gbq(b\tilde{k}) - \frac{1}{2} g^2 q^2 (b\tilde{k}) - 2(N-2)q^2 \xi_9 - 2n_7(b\tilde{k})q^2 + (N-3)gq(b+gq)\xi_9 - g^2 B(b\tilde{k}) \quad (\text{D.27})$$

and

$$M_9 = 2 \left[(N-2)\xi_9 - (b\tilde{k}) \right] q^2 - 2(N-1)b^2(b\tilde{k}) - Ngq(b\tilde{k})b - g^2(b\tilde{k})q^2$$

$$+ 2(N-1)(b\tilde{k})B - (N-1)(N-2)B\xi_9$$

$$- \frac{1}{2} (N-2)g^2 q^2 (b\tilde{k}) - (N-2)gq(b+gq)\xi_9 - (N-2)gbq(b\tilde{k}).$$

$$= 2(N-2)\xi_9 q^2 - 2(N-2)b^2(b\tilde{k}) - (N-2)gbq(b\tilde{k})$$

$$+ 2(N-2)(b\tilde{k})(b^2 + gbq + q^2) - (N-1)(N-2)B\xi_9$$

$$- \frac{1}{2} Ng^2 q^2 (b\tilde{k}) - (N-2)gq(b+gq)\xi_9 - (N-2)gbq(b\tilde{k}),$$

or

$$M_9 = 2(N-2)\xi_9 q^2 - (N-1)(N-2)B\xi_9 + 2 \left(N-2 - \frac{N}{4} g^2 \right) q^2 (b\tilde{k}) - (N-2)gq(b+gq)\xi_9. \quad (\text{D.28})$$

Now we get

$$\rho^i_k = \frac{1}{2K^2} M_7 \delta_k^i + \frac{1}{2K^2} M_8 \frac{1}{A_l A^l} A^i A_k - \frac{2q}{NKg} \left(Y_{\{y\}} y_k A^i + Z_{\{e\}} A_k y^i \right) + \frac{1}{2K^2} M_{10} \frac{1}{K^2} y^i y_k, \quad (\text{D.29})$$

where

$$M_{10} = M_9 - M_7.$$

We have

$$M_{10} = 2(N-2)\xi_9 q^2 - (N-1)(N-2)B\xi_9 + 2 \left(N-2 - \frac{N}{4} g^2 \right) q^2 (b\tilde{k}) - (N-2)gq(b+gq)\xi_9$$

$$-2(N-2)(b\tilde{k})B + (N-3)(N-2)B\xi_9$$

$$+ \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) + (N-3)gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}),$$

or

$$M_{10} = -2(N-2)(b\tilde{k})(B-q^2) - 2(N-2)(B-q^2)\xi_9$$

$$- \frac{3}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}). \quad (\text{D.30})$$

Now we may write:

$$\rho^{ik} = \frac{1}{2K^2}M_7g^{ik} + \frac{1}{2K^2}M_8\frac{1}{A_lA^l}A^iA^k - \frac{2q}{NKg}\frac{1}{K^2}\left(Y_{\{y\}}y^kA^i + Z_{\{e\}}A^ky^i\right) + \frac{1}{2K^4}M_{10}y^iy^k. \quad (\text{D.31})$$

Using in (D.31) the representations

$$g^{ik} = \left[a^{ik} + \frac{g}{q}(bb^ib^k - b^iy^k - b^ky^i) + \frac{g}{Bq}(b+gq)y^iy^k\right]\frac{B}{K^2} \quad (\text{D.32})$$

and

$$A^i = \frac{N}{2}g\frac{1}{qK}\left[Bb^i - (b+gq)y^i\right], \quad (\text{D.33})$$

we arrive at the expansion

$$\rho^{ik} = E_1a^{ik} + E_2b^ib^k + E_{3Y}b^iy^k + E_{3Z}b^ky^i + E_4y^iy^k, \quad (\text{D.34})$$

where

$$E_1 = \frac{B}{2K^4}M_7, \quad (\text{D.35})$$

$$E_2 = \frac{B}{2K^4q^2}(gbqM_7 + BM_8), \quad (\text{D.36})$$

$$E_{3Y} = -\frac{B}{2K^4}\left(\frac{g}{q}M_7 + \frac{1}{q^2}(b+gq)M_8 + 2Y_{\{y\}}\right), \quad (\text{D.37})$$

$$E_{3Z} = -\frac{B}{2K^4}\left(\frac{g}{q}M_7 + \frac{1}{q^2}(b+gq)M_8 + 2Z_{\{e\}}\right), \quad (\text{D.38})$$

$$E_4 = \frac{1}{2K^4}\left[\frac{g}{q}(b+gq)M_7 + \frac{1}{q^2}(b+gq)^2M_8 + 2(b+gq)(Y_{\{y\}} + Z_{\{e\}}) + M_{10}\right]. \quad (\text{D.39})$$

Let us perform the required calculation:

$$-\frac{2}{B}K^4E_{3Y} = \frac{g}{q}\left(2B\mathcal{P} - \frac{N}{2}g^2B(b\tilde{k})\right)$$

$$\begin{aligned}
& -\frac{1}{2}(N-3)g^2q^2(b\tilde{k}) - (N-3)gq(b+gq)\xi_9 - (N-3)gbq(b\tilde{k}) \Big) \\
& + \frac{b}{q^2} \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 - 2(N-2)q^2\xi_9 \right. \\
& \quad \left. - 2n_7(b\tilde{k})q^2 + (N-2)gq(b+gq)\xi_9 - g^2B(b\tilde{k}) \right] \\
& + g\frac{1}{q} \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 - 2(N-2)q^2\xi_9 \right. \\
& \quad \left. - 2n_7(b\tilde{k})q^2 + (N-2)gq(b+gq)\xi_9 - g^2B(b\tilde{k}) \right] \\
& + 2(N-2)b\xi_9 + 2n_7(b\tilde{k})b + (N-2)gq\xi_9.
\end{aligned}$$

Here, all the terms which are proportional to g are cancelled, leaving us with

$$E_{3Y} = gT_{3Y}, \quad (\text{D.40})$$

where

$$\begin{aligned}
-\frac{2}{B}K^4T_{3Y} &= \frac{1}{q} \left(2B\mathcal{P} - \frac{N}{2}g^2B(b\tilde{k}) - \frac{1}{2}(N-2)g^2q^2(b\tilde{k}) - (N-2)gbq(b\tilde{k}) \right) \\
&+ \frac{b}{q} \left[-b(b\tilde{k}) - \frac{1}{2}gq(b\tilde{k}) - (b+gq)\xi_9 + (N-2)(b+gq)\xi_9 - \frac{1}{q}gB(b\tilde{k}) \right] \\
&+ \frac{1}{q} \left[-(N-2)q^2\xi_9 - 2 \left(N-2 + \frac{N}{4}g^2 \right) (b\tilde{k})q^2 - g^2B(b\tilde{k}) \right]. \quad (\text{D.41})
\end{aligned}$$

Also,

$$\begin{aligned}
-\frac{2}{B}K^4E_{3Z} &= \frac{g}{q} \left(2B\mathcal{P} - \frac{N}{2}g^2B(b\tilde{k}) \right. \\
&\quad \left. - \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) - (N-3)gq(b+gq)\xi_9 - (N-3)gbq(b\tilde{k}) \right) \\
&+ \frac{b}{q^2} \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 - 2(N-2)q^2\xi_9 \right.
\end{aligned}$$

$$-2 \left(N - 2 + \frac{N}{4} g^2 \right) (b\tilde{k})q^2 + (N - 2)gq(b + gq)\xi_9 - g^2 B(b\tilde{k}) \Bigg]$$

$$+ g \frac{1}{q} \left[-gbq(b\tilde{k}) - \frac{1}{2} g^2 q^2 (b\tilde{k}) - gq(b + gq)\xi_9 - 2(N - 2)q^2 \xi_9 \right.$$

$$\left. -2n_7(b\tilde{k})q^2 + (N - 2)gq(b + gq)\xi_9 - g^2 B(b\tilde{k}) \right]$$

$$+ 2(N - 2)\xi_9(b + gq) + 2(N - 2)b(b\tilde{k}) + (N - 2)gq(b\tilde{k}) - g^2(b\tilde{k})(b + gq),$$

so that

$$E_{3Z} = gT_{3Z} \tag{D.42}$$

with

$$\begin{aligned} -\frac{2}{B}K^4T_{3Z} &= \frac{1}{q} \left(2B\mathcal{P} - \frac{N}{2}g^2B(b\tilde{k}) - \frac{1}{2}(N - 2)g^2q^2(b\tilde{k}) - (N - 2)gbq(b\tilde{k}) \right) \\ &+ \frac{b}{q} \left[-b(b\tilde{k}) - \frac{1}{2}gq(b\tilde{k}) - (b + gq)\xi_9 - \frac{N}{2}gq(b\tilde{k}) + (N - 2)(b + gq)\xi_9 - \frac{1}{q}gB(b\tilde{k}) \right] \\ &- \frac{1}{q} \left[2 \left(N - 2 + \frac{N}{4}g^2 \right) (b\tilde{k})q^2 + g^2B(b\tilde{k}) \right] + (N - 2)q(b\tilde{k}) - g(b\tilde{k})(b + gq). \end{aligned} \tag{D.43}$$

We can infer that

$$\frac{2}{B}K^4(T_{3Y} - T_{3Z}) = T_7, \tag{D.44}$$

where T_7 is just the function (E.2) of Appendix E.

We continue:

$$\begin{aligned} 2K^4E_4 &= \frac{g}{q}(b + gq) \left(2(N - 2)(b\tilde{k})B - (N - 3)(N - 2)B\xi_9 \right. \\ &\quad \left. - \frac{1}{2}(N - 3)g^2q^2(b\tilde{k}) - (N - 3)gq(b + gq)\xi_9 - (N - 3)gbq(b\tilde{k}) \right) \\ &+ \frac{1}{q^2}(b + gq)^2 \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b + gq)\xi_9 - 2(N - 2)q^2\xi_9 \right. \end{aligned}$$

$$\begin{aligned}
& \left[-2n_7(b\tilde{k})q^2 + (N-2)gq(b+gq)\xi_9 - g^2B(b\tilde{k}) \right] \\
& + 2(b+gq) \left[(N-2)b\xi_9 + n_7(b\tilde{k})b + \frac{1}{2}(N-2)gq\xi_9 \right] \\
& + 2(b+gq) \left[(N-2)\xi_9(b+gq) + (N-2)b(b\tilde{k}) + \frac{1}{2}(N-2)gq(b\tilde{k}) - \frac{1}{2}g^2(b\tilde{k})(b+gq) \right] \\
& - 2(N-2)(b\tilde{k})(B-q^2) - 2(N-2)(B-q^2)\xi_9 \\
& - \frac{3}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}),
\end{aligned}$$

so that

$$E_4 = gT_4 \tag{D.45}$$

and

$$\begin{aligned}
& 2K^4T_4 = \frac{2}{q}(b+gq)B\mathcal{P} \\
& -\frac{b}{q} \left(\frac{N}{2}g^2B(b\tilde{k}) + \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) + (N-3)gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}) \right) \\
& -g \left(\frac{N}{2}g^2B(b\tilde{k}) + \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) + (N-3)gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}) \right) \\
& + \frac{b^2}{q^2} \left[-bq(b\tilde{k}) - \frac{3}{2}gq^2(b\tilde{k}) - q(b+gq)\xi_9 + (N-2)q(b+gq)\xi_9 - gB(b\tilde{k}) \right] \\
& + 2\frac{b}{q} \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 - 2(N-2)q^2\xi_9 \right. \\
& \left. - 2n_7(b\tilde{k})q^2 + (N-2)gq(b+gq)\xi_9 - g^2B(b\tilde{k}) \right]
\end{aligned}$$

$$\begin{aligned}
& +g \left[-gbq(b\tilde{k}) - \frac{1}{2}g^2q^2(b\tilde{k}) - gq(b+gq)\xi_9 - 2(N-2)q^2\xi_9 \right. \\
& \quad \left. - 2n_7(b\tilde{k})q^2 + (N-2)gq(b+gq)\xi_9 - g^2B(b\tilde{k}) \right] \\
& + (N-2)bq\xi_9 + 2q \left[(N-2)b\xi_9 + n_7(b\tilde{k})b + \frac{1}{2}(N-2)gq\xi_9 \right] \\
& + 2(N-2)bq\xi_9 + (N-2)bq(b\tilde{k}) + 2q \left[(N-2)gq\xi_9 + \frac{1}{2}(N-2)gq(b\tilde{k}) \right] \\
& - \frac{3}{2}gq^2(b\tilde{k}) - q(b+gq)\xi_9 + (N-3)bq(b\tilde{k}).
\end{aligned}$$

We can simplify as follows:

$$\begin{aligned}
& 2K^4T_4 = \frac{2}{q}(b+gq)B\mathcal{P} \\
& -\frac{b}{q} \left(\frac{N}{2}g^2B(b\tilde{k}) + \frac{1}{2}(N-3)g^2q^2(b\tilde{k}) + (N-3)gq(b+gq)\xi_9 + (N-3)gbq(b\tilde{k}) \right) \\
& -g \left(\frac{N}{2}g^2B(b\tilde{k}) + \frac{1}{2}(N-2)g^2q^2(b\tilde{k}) + (N-3)gbq(b\tilde{k}) \right) \\
& + \frac{b^2}{q^2} \left[-bq(b\tilde{k}) - \frac{3}{2}gq^2(b\tilde{k}) + (N-3)q(b+gq)\xi_9 - gB(b\tilde{k}) \right] \\
& + \frac{b}{q} \left[-2gbq(b\tilde{k}) - 3g^2q^2(b\tilde{k}) + 2(N-3)gq(b+gq)\xi_9 - 2g^2B(b\tilde{k}) \right] \\
& +g \left[-2 \left(N-2 + \frac{N}{4}g^2 \right) (b\tilde{k})q^2 - g^2B(b\tilde{k}) \right] \\
& + (N-2)bq\xi_9 + (N-2)gq^2\xi_9 + (N-2)bq(b\tilde{k}) + (N-2)gq^2(b\tilde{k}) \\
& - \frac{3}{2}gq^2(b\tilde{k}) - q(b+gq)\xi_9 + (N-3)bq(b\tilde{k}).
\end{aligned} \tag{D.46}$$

In (D.20) we may apply the formula

$$A^i = \frac{N}{2}g\frac{1}{qK}\left[Bb^i - (b + gq)y^i\right]$$

(see (D.33)) and obtain

$$\rho^i_k = \frac{1}{2K^2}M_7\left[h_k^i - \frac{1}{A_l A^l}A^i A_k\right] - \frac{1}{2}\frac{1}{A_l A^l}A^i A_k R^{nh}_{hn} - \frac{B^2}{K^2 q^2}Y_k b^i + \frac{1}{K^2}P_k y^i, \quad (\text{D.47})$$

where

$$P_k = \frac{B}{q^2}(b + gq)Y_k + Z_k - \frac{1}{2}y_k R^{nh}_{hn}. \quad (\text{D.48})$$

We can decompose this vector:

$$P_k = P_{\{y\}}\frac{1}{K^2}y_k + P_{\{e\}}\frac{q^2}{B}e_k \quad (\text{D.49})$$

and use (D.7)–(D.12), getting

$$\begin{aligned} P_{\{y\}} &= (b + gq)Y_{\{y\}} + Z_{\{y\}} - \frac{1}{2}K^2 R^{nh}_{hn} \\ &= b\left((N - 2)b\xi_9 + n_7(b\tilde{k})b + \frac{1}{2}(N - 2)gq\xi_9\right) \\ &\quad + gq\left((N - 2)b\xi_9 + n_7(b\tilde{k})b + \frac{1}{2}(N - 2)gq\xi_9\right) \\ &\quad + \left[(N - 2)\xi_9 - (b\tilde{k})\right]q^2 - (N - 1)b^2(b\tilde{k}) - \frac{N}{2}gbq(b\tilde{k}) - \frac{1}{2}g^2(b\tilde{k})q^2 \\ &\quad + (N - 1)(b\tilde{k})B - \frac{1}{2}(N - 1)(N - 2)B\xi_9 \\ &\quad - \frac{1}{4}(N - 2)g^2q^2(b\tilde{k}) - \frac{1}{2}(N - 2)gq(b + gq)\xi_9 - \frac{1}{2}(N - 2)gbq(b\tilde{k}). \end{aligned}$$

Here, several reductions are available, namely,

$$\begin{aligned} P_{\{y\}} &= n_7(b\tilde{k})b^2 + n_7(b\tilde{k})gbq - q^2(b\tilde{k}) - (N - 1)b^2(b\tilde{k}) - \frac{N}{2}gbq(b\tilde{k}) \\ &\quad + (N - 1)(b\tilde{k})B - \frac{1}{2}(N - 3)(N - 2)B\xi_9 - \frac{1}{4}Ng^2q^2(b\tilde{k}) - \frac{1}{2}(N - 2)gbq(b\tilde{k}) \\ &= \left(N - 2 + \frac{N}{4}g^2\right)(b\tilde{k})b^2 + \left(N - 2 + \frac{N}{4}g^2\right)(b\tilde{k})gbq \end{aligned}$$

$$\begin{aligned}
& -(N-2)b^2(\tilde{b}\tilde{k})(\tilde{b}\tilde{k}) + (N-2)(\tilde{b}\tilde{k})B - \frac{1}{2}(N-3)(N-2)B\xi_9 - \frac{1}{4}Ng^2q^2(\tilde{b}\tilde{k}) - (N-2)gbq(\tilde{b}\tilde{k}) \\
& = (N-2)(\tilde{b}\tilde{k})B - \frac{1}{2}(N-2)(N-3)B\xi_9 + \frac{1}{4}Ng^2(\tilde{b}\tilde{k})(b^2 + gbq) - \frac{1}{4}Ng^2q^2(\tilde{b}\tilde{k}).
\end{aligned}$$

Eventually, we arrive at the function

$$P_{\{y\}} = n_7(\tilde{b}\tilde{k})B - \frac{1}{2}(N-2)(N-3)B\xi_9 - \frac{1}{2}Ng^2q^2(\tilde{b}\tilde{k}) \quad (\text{D.50})$$

which is tantamount to the function (C.62) of Appendix C:

$$P_{\{y\}} = P. \quad (\text{D.51})$$

Appendix E: Skew part

In the special case of Appendix C from (B.53) we can find the skew part $\rho_{ik} - \rho_{ki}$ of the covariantly conserved tensor ρ_{ik} :

$$\begin{aligned}
\rho_{ik} - \rho_{ki} &= \left[(N-2)\frac{g}{B}q\frac{1}{4}k^2g^2 - \frac{1}{2}N\frac{g^2b}{B}(\tilde{b}\tilde{k}) \right] (b_iv_k - b_kv_i) \\
&+ \frac{g}{B}\left(qb_i - b\frac{v_i}{q}\right)y^ja_{njkm}a^{nm} - \frac{g}{B}\left(qb_k - b\frac{v_k}{q}\right)y^ja_{njim}a^{nm} \\
&+ \frac{g}{Bq}(b + gq)y^my^n(a_{nikm} - a_{nkim}) \\
&+ (N-1)\frac{g}{Bq}(q^2 + b^2)(\tilde{b}\tilde{k})(b_iv_k - b_kv_i) \\
&- \frac{g}{q}b\frac{g}{B}q(\tilde{b}\tilde{k})(b_iv_k - b_kv_i) - \frac{g}{q}\frac{g^2}{B}q^2(\tilde{b}\tilde{k})(b_iv_k - b_kv_i).
\end{aligned}$$

Using here the representations (C.42) and (C.43) specific for the isotropic case yields

$$\begin{aligned}
\rho_{ik} - \rho_{ki} &= \frac{1}{4}(N-2)\frac{g}{B}qk^2g^2(b_iv_k - b_kv_i) - \frac{1}{2}N\frac{g^2}{B}b(\tilde{b}\tilde{k})(b_iv_k - b_kv_i) \\
&- \frac{g}{B}\left(qb_i - b\frac{v_i}{q}\right)\left((N-1)(\tilde{b}\tilde{k})bb_k - (N-2)\xi v_k + (\tilde{b}\tilde{k})v_k\right) \\
&+ \frac{g}{B}\left(qb_k - b\frac{v_k}{q}\right)\left((N-1)(\tilde{b}\tilde{k})bb_i - (N-2)\xi v_i + (\tilde{b}\tilde{k})v_i\right) \\
&+ (N-1)\frac{g}{Bq}(q^2 + b^2)(\tilde{b}\tilde{k})(b_iv_k - b_kv_i)
\end{aligned}$$

$$-\frac{g^2(b+gq)}{B}(b\tilde{k})(b_iv_k-b_kv_i),$$

or

$$\begin{aligned}\rho_{ik}-\rho_{ki} &= \frac{1}{4}(N-2)\frac{g}{B}qk^2g^2(b_iv_k-b_kv_i)-\frac{1}{2}N\frac{g^2}{B}b(b\tilde{k})(b_iv_k-b_kv_i) \\ &\quad -\frac{gq}{B}\left(-(N-2)\xi+(b\tilde{k})\right)(b_iv_k-b_kv_i)-(N-1)\frac{gb^2}{qB}(b\tilde{k})(b_iv_k-b_kv_i) \\ &\quad +(N-1)\frac{g}{Bq}(q^2+b^2)(b\tilde{k})(b_iv_k-b_kv_i)-\frac{g^2(b+gq)}{B}(b\tilde{k})(b_iv_k-b_kv_i) \\ &= -\frac{1}{2}N\frac{g^2}{B}b(b\tilde{k})(b_iv_k-b_kv_i)-\frac{gq}{B}\left(-(N-2)\xi_9+(b\tilde{k})\right)(b_iv_k-b_kv_i) \\ &\quad +(N-1)\frac{gq}{B}(b\tilde{k})(b_iv_k-b_kv_i)-\frac{g^2(b+gq)}{B}(b\tilde{k})(b_iv_k-b_kv_i).\end{aligned}$$

Eventually, we get

$$\frac{1}{2}(\rho_{ik}-\rho_{ki})=\frac{g}{2B}T_7(b_iv_k-b_kv_i), \quad (\text{E.1})$$

where

$$T_7=-\frac{1}{2}Ngb(b\tilde{k})-g(b+gq)(b\tilde{k})+(N-2)q\left(\xi_9+(b\tilde{k})\right). \quad (\text{E.2})$$

If we now apply the formulas

$$y_i=\left(v_i+(b+gq)b_i\right)\frac{K^2}{B}, \quad A_i=\frac{NK}{2}g\frac{1}{q}\left(b_i-\frac{b}{K^2}y_i\right),$$

we obtain

$$\frac{g}{B}(b_iv_k-b_kv_i)=\frac{g}{K^2}(b_iy_k-b_ky_i)=\frac{g}{K^2}\frac{2}{NK}\frac{q}{g}(A_iy_k-A_ky_i),$$

so that

$$\frac{g}{B}(b_iv_k-b_kv_i)=\frac{q}{K^2}\frac{2}{N}(A_il_k-A_kl_i) \quad (\text{E.3})$$

and we may write (E.1) in the tensorial form

$$\frac{1}{2}(\rho_{ik}-\rho_{ki})=\frac{1}{NK^2}qT_7(A_il_k-A_kl_i). \quad (\text{E.4})$$

From (A.65) we can infer

$$\rho_{ij}-\rho_{ji}=R_i^m{}_{mj}-R_j^m{}_{mi}+R^m{}_{ijm}-R^m{}_{jim}, \quad (\text{E.5})$$

so that

$$A^iy^j(R_i^m{}_{mj}-R_j^m{}_{mi}+R^m{}_{ijm}-R^m{}_{jim})=\frac{4}{N}qT_7A^iA_i, \quad (\text{E.6})$$

or

$$Ng^2qT_7 = A^iy^j(R_i^m{}_{mj} - R_j^m{}_{mi} + R^m{}_{ijm} - R^m{}_{jim}). \quad (\text{E.7})$$

This formula shows how the coefficient T_7 is expressed through the curvature tensor.

Since $(v_k)|_{y^h=b^h} = 0$, from (E.1) and (E.2) we should conclude that

$$(\rho_{ik} - \rho_{ki})|_{y^h=b^h} = 0. \quad (\text{E.8})$$

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